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Study of Fractional Boundary Value Problems

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DEDICATION

I dedicate this achievement to you.

To those of the first and greatest favor after **ALLAH** who accompanied me throughout my journey of knowledge and research, with gratitude for their kindness and in appreciation of their sacrifices.

My dear parents .

To the spring of tenderness and the symbol of giving, whose prayers were the secret of my success and achievement, and whose encouragement was my constant companion throughout my studies.

My dear mother

To my support and my role model, who taught me how to stand tall in the face of a storm.

My dear father

To my support and my companions.

My dear sisters

To the sleepless nights that blossomed into success and to the fatigue that has turned into joy.

I dedicate to you the fruit of my continuous and joint efforts.

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My gratitude to ***My parents*** *is beyond words. I cannot express my appreciation for their sacrifices, prayers, and constant moral support.*

I also thank ***My sisters*** *for their constant support.*

NOTATIONS

$\mathbb{N}$	Set of natural numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^n$	Euclidean space of dimension $n \in \mathbb{N}$.
$x = (x_1, \ldots, x_n)$	An element of $\mathbb{R}^n$.
$\lvert x \rvert$	Euclidean norm in $\mathbb{R}^n$.
Ω	Bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
$\lVert \cdot \rVert_X$	Norm in the Banach space X.
$L^p(\Omega)$	Space of measurable functions such that $\int_\Omega \lvert u \rvert^p \, dx < \infty$, $1 \leq p < \infty$.
$C_0(\Omega)$	Space of continuous functions on Ω vanishing at infinity.
$C^1(\Omega)$	Space of continuously differentiable functions on Ω.
$\partial_t^{\alpha,\eta}$	Fractional time derivative of order $\alpha \in (0,1)$ with parameter $\eta > 0$.

$\mathrm{B}(.)$	Beta function.
$\Gamma(.)$	Gamma function.
I_{a+}^{α}	The fractional integral in the Riemann-Liouville sense of order α to the right.
I_{a-}^{α}	The fractional integral in the Riemann-Liouville sense of order α to the left.
${}^{\rho}I_{\theta}^{\sigma}+\mu$	The left-sided in the generalized fractional integral.
${}_{c}^{\varrho}D_{a}^{\alpha}$	The Caputo-Katugampola fractional derivative.
${}^{\rho}D_{a+}^{\alpha}$	The fractional derivative in the Caputo sense of orderα to the right.
${}^{\rho}D_{a-}^{\alpha}$	The fractional derivative in the Caputo sense of orderα to the left.
$\mathrm{E}_{\alpha,\beta}(z)$	The Mittag-Leffler function.
$\mu(\tau)$	Solution function.
$\mathrm{h}(\tau, r)$	Green's function.

INTRODUCTION

Fractional calculus, a natural extension of classical calculus to non-integer orders, has emerged as a powerful mathematical tool for modeling complex systems exhibiting memory effects, hereditary properties, and long-range interactions. Unlike integer-order derivatives, fractional derivatives capture the past history of a process, making them indispensable in fields such as viscoelasticity, anomalous diffusion, signal processing, control theory, and biology.

Among the various fractional operators, the Caputo–Katugampola derivative stands out due to its ability to generalize both the Caputo and Riemann–Liouville derivatives while incorporating an additional parameter $\rho > 0$. This flexibility allows for a more accurate description of real-world phenomena. However, the analysis of boundary value problems (BVPs) involving such generalized operators remains a challenging area of research.

This thesis focuses on two classes of nonlinear fractional boundary value problems:

1. A two-point BVP involving the Caputo–Katugampola fractional derivative of order $1 < \sigma \leq 2$:

   $$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = -\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{\theta+}\mu(\tau)\right), & \theta < \tau < \vartheta, \\ \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases}$$

   where the nonlinear term Ξ depends on both the unknown function and its fractional derivative.

2. A three-point BVP with Riemann–Liouville fractional derivative of order $\alpha \in (1, 2)$:

   $$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\nu}v(t)), & t \in (0, T), \\ u(0) = 0, \quad u(T) = a\, u(\xi), & \xi \in (0, T), \end{cases}$$

   where $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$.

The study of fractional differential equations has attracted considerable attention in recent decades. Kilbas, Srivastava, and Trujillo [14] provided a comprehensive foundation for the theory and applications of fractional calculus. The Caputo derivative, introduced by Caputo in the 1960s, is widely used due to its ability to incorporate integer-order initial conditions.

Katugampola [9–12] introduced a new family of fractional integrals and derivatives that unify the Riemann–Liouville and Hadamard operators. The Caputo–Katugampola derivative, further developed in [6, 7, 15], has been applied to various boundary value problems.

Existence and uniqueness results for fractional BVPs are often obtained using fixed-point theorems. The Banach contraction principle [2, 13] is frequently employed for uniqueness, while the Leray–Schauder nonlinear alternative [8] and Schaefer's theorem are used for existence results. Green's function techniques play a crucial role in converting fractional differential equations into equivalent integral equations [1].

The main contributions of this work are:

- Derivation of the Green function for the Caputo–Katugampola BVP and establishment of precise bounds for integral kernels.
- Proof of existence and uniqueness of solutions via the Banach contraction principle, with numerical simulations validating the theoretical conditions.
- Construction of the Green function for a three-point Riemann–Liouville fractional BVP and proof of existence of at least one nontrivial solution using the Leray–Schauder nonlinear alternative.
- Presentation of illustrative examples and numerical tables showing the effect of fractional parameters on the solution.

This thesis is structured as follows:

- **Chapter 1** recalls essential definitions and lemmas from fractional calculus, including the Gamma and Beta functions, the Mittag–Leffler function, Riemann–Liouville integrals, Caputo derivatives, and the generalized Katugampola operators. Key fixed-point theorems (Banach contraction, Leray–Schauder) and Hölder's inequality are also presented.
- **Chapter 2** investigates the existence and uniqueness of solutions for a Caputo–Katugampola fractional BVP with two-point boundary conditions. The Green function is derived, and the Banach contraction principle is applied. Numerical examples for different fractional orders σ are provided.
- **Chapter 3** studies the existence of nontrivial solutions for a three-point fractional BVP involving the Riemann–Liouville derivative. The Leray–Schauder nonlinear alternative is employed, and sufficient conditions are established. Several illustrative examples are given.
- **Conclusion** summarizes the main findings and suggests directions for future research.

CHAPTER 1

PRELIMINARIES

[LE]**Notations**

In the preliminaries, we establish the fundamental concepts of fractional calculus that form the basis of our analysis. This includes recalling generalized fractional integrals and derivatives, particularly the **Caputo–Katugampola** operator, which plays a central role in the boundary value problems studied here. By presenting these definitions and essential properties, we provide the mathematical framework necessary for the proofs and results developed in the subsequent sections.

1.1 Special Functions

1.1.1 Gamma Function

Definition 1.1.1 ([14]) *The Gamma function $\Gamma(z)$ is generally defined*

by the following integral:

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \quad \Re(z) > 0. \tag{1.1}$$

This integral is convergent for any complex number $z \in \mathbb{C}$, provided that $\Re(z) > 0$.

Proposition 1.1.1 ([14]) *One of the fundamental properties of the Gamma function is that it satisfies the following functional equation:*

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2}$$

In particular:

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}^*.$$

Another important property of the Gamma function is that it has simple poles at the points $z = -n$, for $(n = 0, 1, 2, \dots)$.

1.1.2 Beta Function

Definition 1.1.2 ([14]) *The Beta function is defined by Euler's integral of the second kind*

$$\mathrm{B}(z,\omega)=\int_0^1 t^{z-1}(1-t)^{\omega-1}dt,\quad \Re(z)>0,\quad \Re(\omega)>0. \tag{1.4}$$

This integral converges for all $z,\omega\in\mathbb{C}$ if $\Re(z)>0$ and $\Re(\omega)>0$.

Proposition 1.1.2 ([14]) *The Beta function is related to the Gamma function by the following relation:*

$$\mathrm{B}(z,\omega)=\frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)},\quad \Re(z)>0,\quad \Re(\omega)>0. \tag{1.5}$$

Hence, it follows that the Beta function is symmetric:

$$\mathrm{B}(z,\omega)=\mathrm{B}(\omega,z),\quad \Re(z)>0,\quad \Re(\omega)>0.$$

Proof. *Let $(z,\omega)\in\mathbb{C}^2$, with $\Re(z)>0$ and $\Re(\omega)>0$. Using Fubini's theorem, we obtain*

$$\Gamma(z)\Gamma(\omega)=\int_0^\infty\int_0^\infty t^{z-1}e^{-t}x^{\omega-1}e^{-x}dtdx.$$

We perform the change of variable $r=t+x$

$$\begin{aligned}\Gamma(z)\Gamma(\omega)&=\int_0^\infty t^{z-1}\left[\int_t^\infty (r-t)^{\omega-1}e^{-r}dr\right]dt\\&=\int_0^\infty e^{-r}\left[\int_0^r t^{z-1}(r-t)^{\omega-1}dt\right]dr.\end{aligned}$$

We set $t=rs$ and obtain

$$\begin{aligned}\Gamma(z)\Gamma(\omega)&=\int_0^\infty e^{-r}\int_0^1 (r-rs)^{\omega-1}(rs)^{z-1}r\,ds\,dr\\&=\int_0^\infty e^{-r}r^{\omega-1}r^{z-1}r\int_0^1(1-s)^{\omega-1}s^{z-1}ds\,dr\\&=\int_0^\infty e^{-r}r^{z+\omega-1}dr\int_0^1(1-s)^{\omega-1}s^{z-1}ds\\&=\Gamma(z+\omega)\,\mathrm{B}(z,\omega).\end{aligned}$$

1.1.3 Mittag-Leffler Function

Definition 1.1.3 ([14]) *The Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C},\ \alpha,\beta>0.$$

For $\beta=1$

$$E_{\alpha,1}(z)=E_\alpha(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad z\in\mathbb{C},\ \alpha>0.$$

And for $\beta=1,\alpha=1$ we have

$$E_{1,1}(z)=e^z.$$

The latter plays a very important role in the theory of integer-order differential equations.

1.2 Caputo Fractional Calculus

1.2.1 Caputo Fractional Derivative

Definition 1.2.1 ([3]) *Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $u \in C([0,1), \mathbf{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

1.2.2 Basic Lemmas for Caputo Derivative

Lemma 1.2.1 *Let $\alpha > 0$, $n-1 < \alpha < n$ and the function $g : [0;T] \longrightarrow \mathbf{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

Lemma 1.2.2 *Assume that $u \in C[0,1] \cap L^1(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I^{\alpha}\ {}^{c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

1.3 Generalized Katugampola Operators

1.3.1 Generalized Fractional Integral

Definition 1.3.1 ([7]) *The left-sided generalized fractional integral ${}^{\varrho}I^{\sigma}_{\theta+}\mu$ of order $\sigma \in \mathbb{C}(Re(\sigma) > 0)$ is defined by*

$$\left({}^{\varrho}I^{\sigma}_{\theta+}\mu\right)(\tau) = \tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)} \int_{\theta}^{\tau} r^{\varrho-1} (\tau^{\varrho} - r^{\varrho})^{\sigma-1} \mu(r)\, \mathrm{d}r, \tag{1.3}$$

where $\tau > 0$, $\varrho > 0$. According to the formula of the generalized fractional integrals (1.3).

1.3.2 Generalized Caputo Non-Classical Derivative

Definition 1.3.2 ([7]) *By using the above Caputo-Katugampola fractional derivative* (1.5)*, the generalized Caputo non-classical derivative with the operator notation ${}^{\varrho}_{c}D^{\sigma}_{\theta+}$ is displayed by*

$${}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = \left(\varrho D^{\sigma}_{\theta+}\left[\mu(\tau) - \sum_{l=0}^{n-1} \tfrac{\mu^{(l)}(\theta)}{l!}(\tau-\theta)^l\right]\right)(\tau), \quad n = [Re(\sigma)]. \tag{1.4}$$

1.3.3 Caputo–Katugampola fractional derivative

Definition 1.3.3 *We define the Caputo-Katugampola fractional derivative for $\tau > 0$ by*

$$\begin{aligned}\left({}^{\varrho}D^{\sigma}_{\theta+}\mu\right)(\tau) &= \left(\tau^{1-\varrho}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\left({}^{\varrho}I^{n-\sigma}_{\theta+}\mu\right)(\tau)\\ &= \tfrac{\varrho^{\sigma-n+1}}{\Gamma(n-\sigma)}\left(\tau^{1-\varrho}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{n-1-\sigma}\mu(r)\,\mathrm{d}r. \end{aligned} \tag{1.5}$$

1.3.4 Properties and Solutions

Lemma 1.3.1 ([15]) *Let $\sigma, \varrho > 0$ and $\mu \in \mathcal{C}(J,\mathbb{R})\cap\mathcal{C}^{1}(J,\mathbb{R})$. The Caputo-Katugampola fractional derivative differential equation*

$${}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = 0,$$

has a solution.

$$\mu(\tau) = p_0 + p_1\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{2} + \ldots + p_{n-1}\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1},$$

where $p_i \in \mathbb{R}$, $\quad i = 0,1,2,\ldots,n-1$ and $n = [\sigma]+1$.

2. *If $\quad \mu,\ {}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu \in \mathcal{C}(J,\mathbb{R})\cap\mathcal{C}^{1}(J,\mathbb{R})$. Then*

$${}^{\varrho}I^{\sigma}_{\theta+}\,{}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = \mu(\tau) + p_0 + p_1\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{2} + \ldots + p_{n-1}\left(\tfrac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1}, \tag{1.6}$$

where $p_i \in \mathbb{R}$, $\quad i = 0,1,2,\ldots,n-1$ and $n = [\sigma]+1$.

1.4 Riemann-Liouville Fractional

1.4.1 Riemann-Liouville fractional integral

Definition 1.4.1 ([3]) *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \longrightarrow \mathbf{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

1.4.2 Riemann-Liouville fractional derivative

Definition 1.4.2 ([3]) *Let $\alpha > 0$ and let $n = \lceil\alpha\rceil$ be the smallest integer greater than or equal to α (i.e., $n-1 < \alpha \le n$). The Riemann-Liouville fractional derivative of order α of a function $u : (0,\infty) \longrightarrow \mathbb{R}$ is defined by*

$$D^{\alpha}u(t) = \frac{d^n}{dt^n}\left(I^{n-\alpha}u(t)\right),$$

or equivalently,

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 1.4.1 *([15]) Let $\alpha, \beta > 0$ and $u \in L^p(0,1)$, $1 \le p \le +\infty$. Then the next formulas hold.*
(i) $(I^\beta I^\alpha u)(t) = I^{\alpha+\beta} u(t)$,
(ii) $(D^\beta I^\alpha u)(t) = I^{\alpha-\beta} u(t)$,
(iii) $(D^\alpha I^\alpha u)(t) = u(t)$.

Lemma 1.4.2 *For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha u(t) = 0$ with $u \in C[0,1] \cap L^1(0,1)$ is given by*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

where $c_i \in \mathbf{R}$, $\quad i = 1, 2, ..., n$. Hence for $u \in C[0,1] \cap L^1(0,1)$, we have

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}.$$

1.5 Functional Analysis Theorems

1.5.1 Operator Completely Continuous

Theorem 1.5.1 *Let $f : X \to X$ be completely continuous. Then either there exists for each $\lambda \in [0,1]$ at least one $x \in X$ such that $x = \lambda f(x)$, or the fixed point set*

$$\{x \in X : x = \lambda f(x), 0 < \lambda < 1\}$$

is unbounded in X.

1.5.2 Banach Contraction Principle

Definition 1.5.1 *[2] A mapping $T : (X,d) \to (X,d)$ is called a Banach contraction if there exists $\alpha \in (0,1)$ such that*

$$d(Tx, Ty) \le \alpha d(x,y), \quad \forall x, y \in X.$$

1.5.3 Nonlinear Alternative of Leray-Schauder

Theorem 1.5.2 ([8]) *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $F : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*
(i) *there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $F(u) = \lambda u$, or*
(ii) *there exists a fixed point $u^* \in \overline{\Omega}$ of F.*

1.5.4 Kelley's Theorems for Second-Order BVPs

Theorem 1.5.3 ([6, 13]) *Presume $\Xi : [\theta, \vartheta] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function and satisfies a uniform Lipschitz condition with respect to μ and μ'*

$$\Big| \Xi(\tau, \mu, \mu') - \Xi(\tau, \nu, \nu') \Big| \le \zeta |\mu - \nu| + \eta |\mu' - \nu'|, \tag{1.7}$$

for (τ, μ, μ'), $(\tau, \nu, \nu') \in [\theta, \vartheta] \times \mathbb{R}^2$, where $\zeta \ge 0$ and $\eta > 0$ are reals. If

$$\zeta \tfrac{(\vartheta - \theta)^2}{8} + \eta \tfrac{(\vartheta - \theta)}{2} < 1. \tag{1.8}$$

Then the boundary value problem

$$\mu'' = -\Xi\left(\tau, \mu, \mu'\right), \quad \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \tag{1.9}$$

admits a unique solution.

Theorem 1.5.4 ([7, 13]) *Presume $\Xi : [\theta, \vartheta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and satisfies a uniform Lipschitz condition with respect to μ*

$$\Big|\Xi\left(\tau, \mu\right) - \Xi\left(\tau, \nu\right)\Big| \leq \zeta|\mu - \nu|, \tag{1.10}$$

for (τ, μ), $(\tau, \nu) \in [\theta, \vartheta] \times \mathbb{R}$, where $\zeta > 0$ is real. If

$$\zeta \tfrac{(\vartheta-\theta)^2}{8} < 1. \tag{1.11}$$

Then the boundary value problem

$$\mu'' = -\Xi\left(\tau, \mu\right), \quad \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \tag{1.12}$$

admits a unique solution.

Theorem 1.5.5 (Hölder and Minkowski Inequalities) *Let p and q be two conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) with $1 \leq p \leq \infty$. Let $f, g \in \mathcal{R}([a, b])$. Then we have the following inequalities:*

(i) ***(Hölder's Inequality)***

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}$$

(ii) ***(Minkowski's Inequality)***

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}$$

Definition 1.5.2 (Green's Function) *Let $\mathcal{L}$ be a linear differential operator of order n on $[a, b]$ with homogeneous boundary conditions. A function $G(t, s)$ is called the* ***Green's function*** *if for every continuous f, the unique solution of*

$$\mathcal{L}[u](t) = f(t), \qquad t \in (a, b),$$

satisfying the boundary conditions is given by

$$u(t) = \int_a^b G(t, s)\, f(s)\, ds.$$

Moreover, $G(t, s)$ satisfies:

- *$\mathcal{L}[G(\cdot, s)](t) = 0$ for $t \neq s$,*
- *$G(t, s)$ and its first $n - 2$ derivatives are continuous in t at $t = s$,*
- *The $(n - 1)$-th derivative has a jump of $\dfrac{1}{a_n(s)}$ at $t = s$,*
- *$G(t, s)$ satisfies the homogeneous boundary conditions in t.*

CHAPTER 2

EXISTENCE AND UNIQUENESS OF SOLUTION FOR CAPUTO–KATUGAMPOLA BOUNDARY VALUE PROBLEMS

2.1 Introduction

In this chapter, we achieve the existence of a unique solution to the BVP of the Caputo-Katugampola fractional derivative order

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = -\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{\theta+}\mu(\tau)\right), & \theta < \tau < \vartheta, \\ \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases} \tag{2.1}$$

where $1 < \sigma \leq 2$, $0 < \varsigma \leq 1$. *Previous studies investigated the existence of solutions for boundary value problems involving the Caputo–Katugampola fractional derivative (see, [5]) and the references mentioned in its content). Under previous research, the simulation of our problem produces new results by applying Theorem 1.5.4.*

2.2 Green Function and Integral Equation Formula

In this section, we present the main propositions and theorems used throughout this chapter on which all this work is based. We give the integral formula for the generalized fractional BVP (2.1) *from the principle of the Green function.*

Lemma 2.2.1 *Presume that* Ξ *is a function is continuous and either a function* $\mu \in C[\theta, \vartheta]$ *is a solution of* (2.1) *equivalent that* μ *check the integral equation*

$$\mu(\tau) = \left[(\lambda_2 - \lambda_1)\tfrac{(\tau^{\varrho} - \theta^{\varrho})}{(\vartheta^{\varrho} - \theta^{\varrho})} + \lambda_1\right] + \int_{\theta}^{\vartheta} \hbar(\tau, r)\ \Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{\theta+}\mu(r)\right) \mathrm{d}r, \tag{2.2}$$

where for $r, \tau \in [\theta, \vartheta]$,

$$\hbar(\tau, r) = \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)} \begin{cases} \frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} \quad r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1} - r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}, & r \leq \tau, \\ \frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} \quad r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1}, & \tau \leq r. \end{cases} \tag{2.3}$$

Proof 2.2.1 *By the Lemma(1.3.1),we solve this problem*

$${}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = -q(\tau).$$

According to (1.3.1) we obtain

$$\begin{aligned} {}^{\varrho}I^{\sigma}_{\theta+} \; {}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) &= -{}^{\varrho}I^{\sigma}_{\theta+}q(\tau) + p_0 + p_1\frac{(\tau^{\varrho} - \theta^{\varrho})}{\varrho} \\ \mu(\tau) &= -{}^{\varrho}I^{\sigma}_{\theta+}q(\tau) + p_0 + p_1\frac{(\tau^{\varrho} - \theta^{\varrho})}{\varrho} \\ \mu(\tau) &= -\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}q(r)dr + p_0 + p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho}, \end{aligned}$$

by using boundary conditions

$$\begin{aligned} &\mu(\theta) = \lambda_1 \Longrightarrow p_0 = \lambda_1, \\ &\mu(\vartheta) = \lambda_2 \Longrightarrow p_1 = \frac{\varrho(\lambda_2-\lambda_1)}{(\vartheta^{\varrho}-\theta^{\varrho})} + \frac{\varrho^{2-\sigma}}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)}\int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r. \end{aligned}$$

By replacing in $\mu(\tau)$, we get

$$\begin{aligned} \mu(\tau) = &-\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r \\ &+ \left[\varrho(\lambda_2 - \lambda_1) + \frac{\varrho^{2-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1}q(r)dr\right]\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1, \end{aligned}$$

and

$$\begin{aligned} \mu(\tau) = -\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r + \frac{\varrho^{1-\sigma}(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)} \\ \int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r + (\lambda_2 - \lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1. \end{aligned}$$

Therefore,

$$\begin{aligned} \mu(\tau) = &\left[(\lambda_2 - \lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1\right] \\ &+ \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}\left(\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1} - r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}\right)q(r)\,\mathrm{d}r \\ &+ \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\tau}^{\vartheta}\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho} - r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r, \end{aligned}$$

Thus, the proof is complete.

2.3 Existence and Uniqueness of the Solution

Immediately, we will present the important salient rules that will make it easier for us to achieve your desired objectives.

Proposition 2.3.1 *Depending on the Green function, $\hbar$ is given in Lemma 2.2.1. Therefore*

$$\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r \leq \tfrac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\right], \tag{2.4}$$

and

$$\int_{\theta}^{\vartheta} \tfrac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \leq \tfrac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}\right]. \tag{2.5}$$

Proof 2.3.1 *We determine*

$$\int_{\vartheta}^{\theta} |\hbar(\tau,r)|\,\mathrm{d}r, \qquad \hbar(\tau,r)\geq 0. \quad \forall\,\theta\leq\tau,\ r\leq\vartheta. \tag{2.6}$$

Therefore

$$\begin{aligned}\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r = \tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)} \Bigg[&\int_{\theta}^{\tau} \left(\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}-r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}\right)\mathrm{d}r\\ &+\int_{\tau}^{\vartheta}\left(\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}\right)\mathrm{d}r\Bigg].\end{aligned}$$

We calculate the primitives by integration by a change of variable

$$\begin{aligned}\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r &= \tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\left[\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left[\left(1-\tfrac{\theta^{\varrho}}{\vartheta^{\varrho}}\right)^{\sigma}-\left(1-\tfrac{\tau^{\varrho}}{\vartheta^{\varrho}}\right)^{\sigma}\right]\right.\\ &\quad\left.-\tfrac{\tau^{\varrho\sigma}}{\varrho}\left(\tfrac{1}{\sigma}\left(1-\tfrac{\theta^{\varrho}}{\tau^{\varrho}}\right)^{\sigma}\right)+\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left(1-\tfrac{\tau^{\varrho}}{\vartheta^{\varrho}}\right)^{\sigma}\right]\\ &= \tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\left[\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left[\tfrac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}-\tfrac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\right]\right.\\ &\quad\left.-\tfrac{\tau^{\varrho\sigma}}{\varrho}\left(\tfrac{1}{\sigma}\tfrac{(\tau^{\varrho}-\theta^{\varrho})^{\sigma}}{\tau^{\varrho\sigma}}\right)+\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\tfrac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\right]\\ &= \tfrac{\varrho^{1-\sigma}}{\varrho\sigma\Gamma(\sigma)}\left[\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\right].\end{aligned}$$

Then,

$$\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r = \tfrac{1}{\varrho^{\sigma}\sigma\Gamma(\sigma)}\left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\right].$$

This implies that

$$\int_{\theta}^{\vartheta} \tfrac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r = \tfrac{1}{\varrho^{\sigma-1}\sigma\Gamma(\sigma)}\left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}\right], \tag{2.7}$$

which ends the proof.

Corollary 2.3.1 *We can define the continuous functions ξ and ξ' by*

$$\xi(\tau) = (\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}, \qquad \tau\in[\theta,\vartheta], \tag{2.8}$$

and

$$\xi'(\tau) = (\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}, \quad \tau\in[\theta,\vartheta]. \tag{2.9}$$

Proposition 2.3.2 *By* (2.12) *and* (2.13), *suppose that* $\theta = 0$, $\theta < \vartheta$ *then*

$$\int_0^{\vartheta} |\hbar(\tau, r)| \,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{\sigma-1}}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}} \right], \tag{2.10}$$

and

$$\int_0^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau} \,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[\vartheta^{(\varrho\sigma-1)} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}} - \sigma(\vartheta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}} \right]. \tag{2.11}$$

Proof 2.3.2 *According to Proposition 2.3.1and the Corollary 2.3.1 , we have*

$$\int_{\theta}^{\vartheta} |\hbar(\tau, r)| \,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left(\xi(\tau)\right) \tag{2.12}$$

$$\int_{\theta}^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau} \,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left(\xi'(\tau)\right) \tag{2.13}$$

We differentiating the functions ξ, ξ'

$$\xi'(\tau) = (\vartheta^{\varrho})^{\sigma-1}\varrho(\tau^{\varrho-1}) - \sigma(\tau^{\varrho})^{\sigma-1}\varrho\tau^{\sigma\varrho-1} \tag{2.14}$$

$$\xi''(\tau) = (\varrho-1)\vartheta^{\varrho(\sigma-1)}\tau^{\varrho-2} - \sigma(\varrho\sigma-1)\tau^{\sigma-2} \tag{2.15}$$

We are looking for the critical points

$$\xi'(\tau) = 0$$

$$\varrho(\tau^{\varrho-1})[(\vartheta^{\varrho\sigma-1}) - \sigma(\tau^{\varrho})^{\sigma-1}] = 0$$

$$(\vartheta^{\varrho\sigma-1}) - \sigma(\tau^{\varrho})^{\sigma-1} = 0$$

$$\tau^* = \frac{\vartheta}{\sigma^{1\varrho(\sigma-1)}}$$

and

$$\xi''(\tau) = 0$$

$$(\varrho-1)\vartheta^{\varrho(\sigma-1)}\tau^{\varrho-2} - \sigma(\varrho\sigma-1)\tau^{\sigma-2} = 0$$

$$\tau_1^* = \vartheta \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{1}{\varrho(\sigma-1)}}.$$

Moreover,

$$\xi(\tau^*) = \frac{1}{\varrho^{\sigma}} \left(\frac{\vartheta^{\varrho\sigma}}{\sigma^{1\sigma-1}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma\sigma-1}} \right),$$
$$\xi'(\tau_1^*) = \frac{1}{\varrho^{\sigma-1}} \left(\vartheta^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1\varrho(\sigma-1)} - \sigma(\vartheta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1\varrho(\sigma-1)} \right),$$

That finishes the proof.

Theorem 2.3.1 *Assume $\Xi : [0,\vartheta]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[0,\vartheta]\times\mathbb{R}^2$ with Lipschitz real ζ, thus,*

$$\Big|\Xi\left(\tau,\mu,\mu'\right)-\Xi(\tau,\nu,\nu')\Big| \leq \zeta|\mu-\nu|+\eta|\mu'-\nu'|, \tag{2.16}$$

for (τ,μ,μ'), $(\tau,\nu,\nu')\in[0,\vartheta]\times\mathbb{R}^2$, where $\eta\geq 0$, $\zeta>0$ are constants. If

$$\begin{aligned}\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[&\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}\\ &-\sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\Bigg]<1.\end{aligned} \tag{2.17}$$

Then the BVP :

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau)=-\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(\tau)\right), \quad 0<\tau<\vartheta,\\ \mu(0)=\lambda_1, \qquad \mu(\vartheta)=\lambda_2,\end{cases} \tag{2.18}$$

admits a single solution.

Proof 2.3.3 *Suppose Π is a space of Banach fitted with continuous applications defined on $[0,\vartheta]$ with the norm*

$$\|\mu\|=\max_{\tau\in[0,\vartheta]}\Big\{\zeta|\mu(\tau)|+\eta|\mu'(\tau)|\Big\}.$$

According to Lemma 2.2.1, we have $\mu\in C[0,\vartheta]$ is a solution of (2.18) equivalent to this is the same as the solving an equation in integral form

$$\mu(\tau)=\left[(\lambda_2-\lambda_1)\left(\tfrac{\tau}{\vartheta}\right)^{\varrho}+\lambda_1\right]+\int_0^{\vartheta}\hbar(\tau,r)\ \Xi\left(r,\mu(r),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(r)\right)\,\mathrm{d}r. \tag{2.19}$$

Define the operator $\Sigma:\Pi\to\Pi$ by

$$\Sigma\mu(\tau)=\left[(\lambda_2-\lambda_1)\left(\tfrac{\tau}{\vartheta}\right)^{\varrho}+\lambda_1\right]+\int_0^{\vartheta}\hbar(\tau,r)\ \Xi\left(r,\mu(r),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(r)\right)\,\mathrm{d}r \tag{2.20}$$

for $\tau\in[0,\vartheta]$. We should interpret that the application Σ admits a single fixed point. Assume $\mu,\nu\in\Pi$. Therefore

$$\begin{aligned}\zeta|\Sigma\mu(\tau)-\Sigma\nu(\tau)| &\leq \int_0^{\vartheta}|\hbar(\tau,r)|\left|\Xi\left(r,\mu(r),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(r)\right)-\Xi\left(r,\nu(r),{}^{\varrho}_{c}D^{\varsigma}_{0+}\nu(r)\right)\right|\,\mathrm{d}r\\ &\leq \int_0^{\vartheta}|\hbar(\tau,r)|\left(\zeta|\mu(\tau)-\nu(\tau)|+\eta|\mu'(\tau)-\nu'(\tau)|\right)\,\mathrm{d}r\\ &\leq \zeta\int_0^{\vartheta}|\hbar(\tau,r)|\,\mathrm{d}r\|\mu-\nu\| \leq \frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]\|\mu-\nu\|,\end{aligned}$$

for $\tau \in [0,\vartheta]$, and similarly,

$$\eta \left|(\Sigma\mu)'(\tau) - (\Sigma\nu)'(\tau)\right| \leq \int_0^\vartheta \tfrac{|\hbar(\tau,r)|}{\partial\tau} \left|\Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(r)\right) - \Xi\left(r, \nu(r), {}^{\varrho}_{c}D^{\varsigma}_{0+}\nu(r)\right)\right| \mathrm{d}r$$
$$\leq \int_0^\vartheta \tfrac{|\hbar(\tau,r)|}{\partial\tau} \left(\zeta|\mu(\tau)-\nu(\tau)| + \eta|\mu'(\tau)-\nu'(\tau)|\right) \mathrm{d}r$$
$$\leq \eta \int_0^\vartheta \tfrac{|\hbar(\tau,r)|}{\partial\tau} \mathrm{d}r\|\mu-\nu\| \leq \tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[\vartheta^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}\right.$$
$$\left. -\sigma(\vartheta)^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]\|\mu-\nu\|, \qquad \tau \in [0,\vartheta].$$

Hence, $\|\Sigma\mu - \Sigma\nu\| \leq \varpi\|\mu-\nu\|$, where

$$\varpi = \tfrac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\tfrac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{\varrho(\sigma-1)}}} - \tfrac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right] + \tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\vartheta^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}\right.$$
$$\left. -\sigma(\vartheta)^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right] < 1.$$

Whether we have martyred Proposition 2.3.2. According to (2.17)*, we extrapolate that Σ is a contracting operator on Π, so, by the theorem of contraction mapping of Banach, we culminate in the possible outcome. This means that we conclude that Σ accepts a single fixed point in $C[0,\vartheta]$, which requires that the BVP* (2.18) *admits a single solution.*

Remark 2.3.1 *We analyze this when taking $\sigma = 2$, $\varsigma = 1$, $\theta = 0$ and $\varrho = 1$ in Theorem 2.3.1, through condition* (2.17)*, we obviously find Theorem 1.5.4 such that*

$$\tfrac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\tfrac{\vartheta^{\varrho\sigma}}{\sigma^{frac1\sigma-1}} - \tfrac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right] + \tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\vartheta^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}} -\sigma(\vartheta)^{\varrho\sigma-1}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]$$

$$= \zeta\tfrac{\vartheta^2}{4\Gamma(3)} + \eta\tfrac{\vartheta}{\Gamma(3)} < 1.$$

Proposition 2.3.3 ([5]) *By* (2.12) *and* (2.13)*, suppose that $\theta = 0$, $\vartheta = 1$ then*

$$\int_0^1 |\hbar(\tau,r)|\,\mathrm{d}r \leq \tfrac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\tfrac{1}{\sigma^{\frac{1}{\varrho(\sigma-1)}}} - \tfrac{1}{\sigma^{\sigma/(\sigma-1)}}\right], \tag{2.21}$$

and

$$\int_0^1 \tfrac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \leq \tfrac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho-1)/\varrho(\sigma-1)} - \sigma\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]. \tag{2.22}$$

Theorem 2.3.2 ([5]) *Assume $\Xi : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function and check a condition of uniform Lipschitz concerning the second variable on $[0,1]\times\mathbb{R}^2$ with Lipschitz real ζ, thus,*

$$\Big|\Xi(\tau,\mu,\mu') - \Xi(\tau,\nu,\nu')\Big| \leq \zeta|\mu-\nu| + \eta|\mu'-\nu'|, \tag{2.23}$$

for $(\tau,\mu,\mu'),\ (\tau,\nu,\nu')\in[0,1]\times\mathbb{R}^2$, *where* $\eta\geq 0,\ \zeta>0$ *are constants. If*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{\sigma-1}}}-\frac{1}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}-\sigma\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]<1, \tag{2.24}$$

Then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau)=-\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(\tau)\right), & 0<\tau<1,\\ \mu(0)=\lambda_1, \quad \mu(1)=\lambda_2, \end{cases} \tag{2.25}$$

has a unique solution.

Proof 2.3.4 *Using the same method to prove Proposition 2.3.3 and Theorem 2.3.2 which are used in Theorem 2.3.1 and Proposition 2.3.2.*

Remark 2.3.2 *The same remark 2.3.1, we notice that when* $\sigma=2,\ \varsigma=1,\ \theta=0,\ \vartheta=1$ *and* $\varrho=1$ *in Theorem 2.3.2, through condition* (2.24), *we obviously find Theorem 1.5.4 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{\sigma-1}}}-\frac{1}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}-\sigma\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]$$
$$=\zeta\frac{1}{4\Gamma(3)}+\eta\frac{1}{\Gamma(3)}<1.$$

Proposition 2.3.4 ([5]) *By* (2.12) *and* (2.13), *suppose that* $\theta<\vartheta$ *then*

$$\int_0^{\vartheta}|\hbar(\tau,r)|\,\mathrm{d}r\leq\frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right], \tag{2.26}$$

and

$$\int_0^{\vartheta}\frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r\leq\frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(\vartheta-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}-\sigma(\vartheta-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]. \tag{2.27}$$

Theorem 2.3.3 ([5]) *Assume* $\Xi:[\theta,\vartheta]\times\mathbb{R}^2\to\mathbb{R}$ *is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on* $[\theta,\vartheta]\times\mathbb{R}^2$ *with Lipschitz real* ζ, *thus,*

$$\Big|\Xi\left(\tau,\mu,\mu'\right)-\Xi\left(\tau,\nu,\nu'\right)\Big|\leq\zeta|\mu-\nu|+\eta|\mu'-\nu'|, \tag{2.28}$$

for $(\tau,\mu,\mu'),\ (\tau,\nu,\nu')\in[\theta,\vartheta]\times\mathbb{R}^2$, *where* $\eta\geq 0,\ \zeta>0$ *are constants. If*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[(\vartheta-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}$$

$$-\sigma(\vartheta-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\Bigg]<1, \tag{2.29}$$

Then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau)=-\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(\tau)\right), & 0<\tau<1,\\ \mu(\theta)=\lambda_1, \quad \mu(\vartheta)=\lambda_2, \end{cases} \tag{2.30}$$

has a unique solution.

Proof 2.3.5 *Using the same method to prove Proposition 2.3.4 and Theorem 2.3.3 which are used in Proposition 2.3.2 and also applies to Theorem 2.3.1.*

Remark 2.3.3 *Same previous notes. We notice them in the general case. We apply them when* $\sigma = 2$, $\theta < \vartheta$ *and* $\varrho = 1$ *on Theorem 2.3.3, through condition* (2.29), *we obviously find Theorem 1.5.4 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(\vartheta-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}\right.$$

$$\left.-\sigma(\vartheta-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]=\zeta\frac{(\vartheta-\theta)^{2}}{4\Gamma(3)}+\eta\frac{(\vartheta-\theta)}{\Gamma(3)}<1.$$

Proposition 2.3.5 ([5]) *By* (2.12) *and* (2.13), *suppose that* $\theta < \vartheta = 1$ *then*

$$\int_0^{\vartheta}|\hbar(\tau,r)|\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right], \tag{2.31}$$

and

$$\int_0^{\vartheta}\frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}-\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]. \tag{2.32}$$

$$\int_0^{\vartheta}\frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}-\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]. \tag{2.33}$$

Theorem 2.3.4 ([5]) *Assume* $\Xi : [\theta,1]\times\mathbb{R}^2 \to \mathbb{R}$ *is a continuous function and check a condition of uniform Lipschitz concerning the second variable on* $[\theta,1]\times\mathbb{R}^2$ *with Lipschitz real* ζ, *thus,*

$$\Big|\Xi\left(\tau,\mu,\mu'\right)-\Xi\left(\tau,\nu,\nu'\right)\Big| \le \zeta|\mu-\nu|+\eta|\mu'-\nu'|, \tag{2.34}$$

for (τ,μ,μ'), $(\tau,\nu,\nu')\in[\theta,1]\times\mathbb{R}^2$, *where* $\eta \ge 0$, $\zeta > 0$ *are constants. If*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}\right.$$

$$\left.-\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\right]<1, \tag{2.35}$$

Then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau) = -\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0+}\mu(\tau)\right), & 0<\tau<1,\\ \mu(\theta)=\lambda_1, \quad \mu(1)=\lambda_2, \end{cases} \tag{2.36}$$

has a unique solution.

Proof 2.3.6 *Using the same method to prove Proposition 2.3.5 and Theorem 2.3.4 which are used in Proposition 2.3.2 and also applies to Theorem 2.3.1.*

Remark 2.3.4 *Same previous notes. We notice them in the general case. We apply them when* $\sigma = 2$, $\theta < \vartheta = 1$ *and* $\varrho = 1$ *on Theorem 2.3.4, through condition* (2.35), *we obviously find Theorem 1.5.4 such that*

$$\frac{\zeta}{\varrho^\sigma\Gamma(\sigma+1)}\left[\frac{(1-\theta^\varrho)^\sigma}{\sigma^{\frac{1}{\sigma-1}}}-\frac{(1-\theta^\varrho)^\sigma}{\sigma^{\frac{\sigma}{\sigma-1}}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho-1}{\varrho(\sigma-1)}}$$

$$-\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\frac{\varrho\sigma-1}{\varrho(\sigma-1)}}\Bigg]=\zeta\frac{(1-\theta)^2}{4\Gamma(3)}+\eta\frac{(1-\theta)}{\Gamma(3)}<1.$$

2.4 Simulation and Examples

To prove the desired results above, we take some applications.

Example 2.4.1 *Extrapolate the following application of BVP*

$$\begin{cases} {}^{1}_{c}D^{\sigma}_{0^+}\mu(\tau)=7-\tau^5-\sin(\mu(\tau))-{}^{1}_{c}D^{1}_{0^+}\cos(\mu(\tau)), & 0<\tau<\vartheta,\\ \mu(0)=2, \quad \mu(\vartheta)=3. \end{cases} \tag{2.37}$$

Set, $\varrho = 1$, $\sigma = \{1.5,\ 1.65, 1.8, 1.95\}$, $\varsigma = 1$, $\theta = 0$ *and*

$$\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{\theta^+}\mu(\tau)\right)=\tau^5-7+\sin(\mu(\tau))+{}^{1}_{c}D^{1}_{0^+}\cos(\mu(\tau)).$$

Here,

$$\begin{aligned}
&\Big|\Xi\left(\tau,\mu,\mu'\right)-\Xi\left(\tau,\nu,\nu'\right)\Big|\\
&\qquad=\Big|\tau^5-7+\sin(\mu)+\cos(\mu')-\Big(\tau^5-7+\sin(\nu)+\cos(\nu')\Big)\Big|\\
&\qquad\le\Big|\sin(\mu)-\sin(\nu)\Big|+\Big|\cos(\mu')-\cos(\nu')\Big|\\
&\qquad\le|\mu-\nu|+\Big|-2\sin\tfrac{\mu'+\nu'}{2}\sin\tfrac{\mu'-\nu'}{2}\Big|\\
&\qquad\le\zeta|\mu-\nu|+\eta|\mu'-\nu'|,
\end{aligned}$$

for (τ,μ,μ'), $(\tau,\nu,\nu')\in[0,\vartheta]\times\mathbb{R}^2$, *where* $\eta = 1 \ge 0$, $\zeta = 2 > 0$. *Moreover, we have*

$$\varpi=\frac{\zeta}{\varrho^\sigma\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/(\sigma-1)}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/(\sigma-1)}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[\vartheta^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho-1)/\varrho(\sigma-1)}$$

$$-\sigma(\vartheta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho\sigma-1)/\varrho(\sigma-1)}\Bigg]\approx\left\{\begin{array}{ll}0.9751, & \sigma=1.50,\\ 0.9188, & \sigma=1.65,\\ 0.8508, & \sigma=1.80,\\ 0.7758, & \sigma=1.95,\end{array}\right\}<1.$$

The curves drawn in Figure 2.1 show how the ϖ *changes for different derivative orders* σ. *The important point is that all of them are less than the line* $y = 1$ *in the interval* $[0,\vartheta]$, *and as the order of the derivative approaches the number one, the parameter* ϖ *decreases, but they are still less than one. These results are shown in Table 2.1. By the applications of Theorem 2.3.1, and the condition* (2.17) *is agreed. Then the BVP* (2.37) *accepts a unique solution.*

Tableau 2.1: Numerical results ϖ in Example 2.4.1 for four values of σ.

τ	ϖ			
	$\sigma = 1.5$	$\sigma = 1.65$	$\sigma = 1.80$	$\sigma = 1.95$
0.00	0.0000	0.0000	0.0000	0.0000
0.05	0.1707	0.0978	0.0555	0.0311
0.10	0.2449	0.1562	0.0986	0.0616
0.15	0.3043	0.2069	0.1391	0.0926
0.20	0.3564	0.2538	0.1786	0.1244
0.25	0.4040	0.2984	0.2177	0.1571
0.30	0.4487	0.3415	0.2568	0.1909
0.35	0.4912	0.3837	0.2960	0.2256
0.40	0.5322	0.4253	0.3355	0.2614
0.45	0.5719	0.4664	0.3753	0.2983
0.50	0.6107	0.5073	0.4156	0.3362
0.55	0.6488	0.5481	0.4564	0.3753
0.60	0.6863	0.5888	0.4978	0.4154
0.65	0.7233	0.6295	0.5397	0.4566
0.70	0.7599	0.6703	0.5822	0.4989
0.75	0.7962	0.7112	0.6254	0.5423
0.80	0.8323	0.7523	0.6691	0.5868
0.85	0.8682	0.7936	0.7136	0.6324
0.90	0.9040	0.8351	0.7586	0.6791
0.95	0.9396	0.8768	0.8044	0.7269
1.00	0.9751	0.9188	0.8508	0.7758

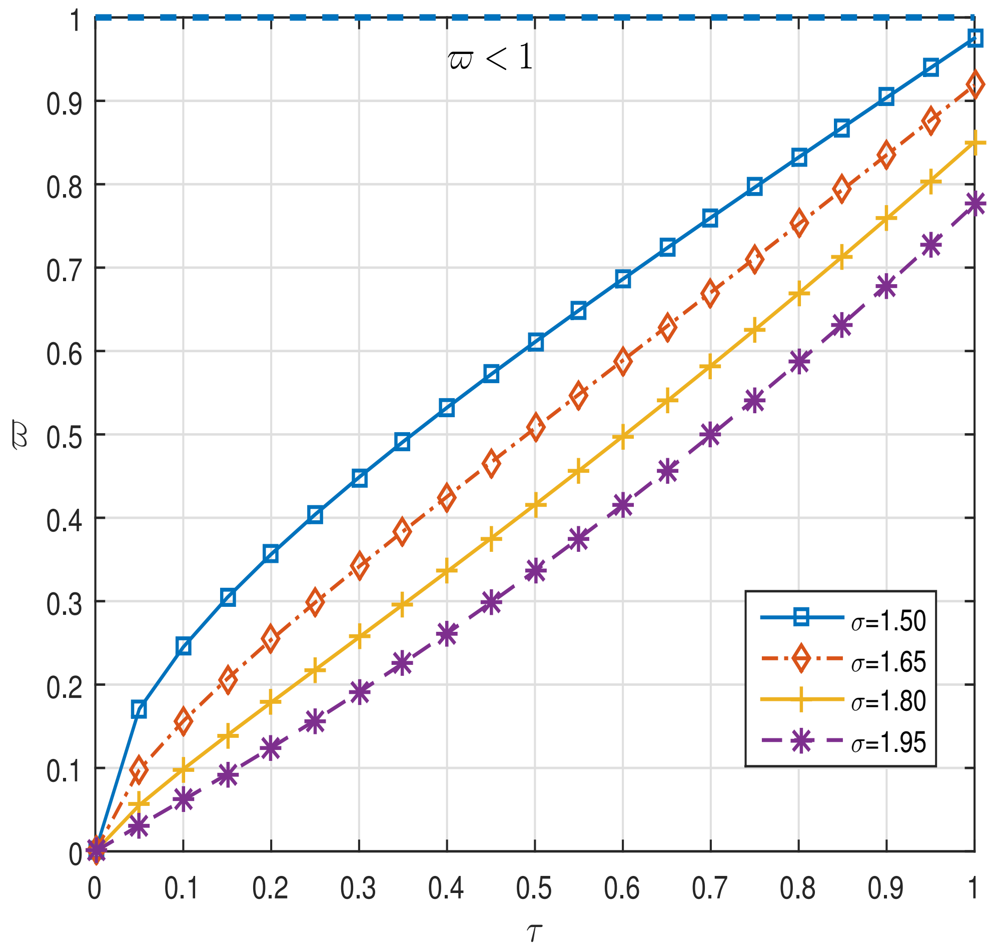

Figure 2.1: Representation of ϖ for BVP (2.37) in Example 2.4.1 for four case σ.

Example 2.4.2 *Extrapolate the following application of BVP*

$$\begin{cases} {}_c^1D^{\sigma}_{0+}\mu(\tau) = 2 - \tau^3 + \frac{1}{2}\cos(\mu(\tau)) - {}_c^1 D^1_{0+}\sin(\mu(\tau)), \quad 0 < \tau < \vartheta \\ \mu(0) = 5, \quad \mu(1) = 6. \end{cases} \tag{2.38}$$

Set, $\varrho = 1$, $\sigma \in \{\frac{4}{3}, \frac{3}{2}, \frac{7}{4}\}$, $\varsigma = 1$, $\theta = 0$, $\vartheta = 1$, *and*

$$\Xi\left(\tau, \mu(\tau), {}_c^{\varrho} D^{\varsigma}_{\theta+}\mu(\tau)\right) = \tau^3 - 2 - \frac{1}{2}\cos(\mu(\tau)) + {}_c^1 D^1_{0+}\sin\left(\mu(\tau)\right).$$

Here,

$$
\begin{aligned}
&\Big|\Xi(\tau,\mu,\mu')-\Xi(\tau,\nu,\nu')\Big|\\
&\qquad=\Big|\tau^3-2-\tfrac{1}{2}\cos(\mu)+\sin(\mu')-\Big(\tau^3-2-\tfrac{1}{2}\cos(\nu)+\sin(\nu')\Big)\Big|\\
&\qquad\le\tfrac{1}{2}\Big|\cos(\nu)-\cos(\mu)\Big|+\Big|\sin(\mu')-\sin(\nu')\Big|\\
&\qquad\le\tfrac{1}{2}\Big|-2\sin\tfrac{\nu+\mu}{2}\sin\tfrac{\nu-\mu}{2}\Big|+|\mu'-\nu'|\\
&\qquad\le\Big|\sin\tfrac{\nu-\mu}{2}\Big|+|\mu'-\nu'|\le\zeta|\mu-\nu|+\eta|\mu'-\nu'|,
\end{aligned}
$$

for (τ,μ,μ'), $(\tau,\nu,\nu')\in[0,1]\times\mathbb{R}^2$, *where* $\eta=1\ge 0$, $\zeta=1>0$. *Moreover, we have*

$$
\begin{aligned}
\varpi=\tfrac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\Big[\tfrac{1}{\sigma^{1/(\sigma-1)}}-\tfrac{1}{\sigma^{\sigma/(\sigma-1)}}\Big]+\tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Big[\Big(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\Big)^{(\varrho-1)/\varrho(\sigma-1)}\\
-\sigma\Big(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\Big)^{(\varrho\sigma-1)/\varrho(\sigma-1)}\Big]\approx\left\{\begin{array}{ll}0.9285, & \sigma=\frac{4}{3},\\ 0.3989, & \sigma=\frac{3}{2},\\ 0.7481, & \sigma=\frac{7}{4},\end{array}\right\}<1.
\end{aligned}
$$

In the last row of data in Table 2.2, the values of parameter ϖ, *at point* ϑ, *for three different values of derivative order* σ *are shown. The curves of all three cases are presented in Figure 2.2, which are decreasing as the order of the derivative increases and in all cases are less than the* $y=1$ *line. By the applications of Theorem 2.3.2, and the condition* (2.24) *is agreed. Then the BVP* (2.38) *accepts a unique solution.*

Tableau 2.2: Numerical results ϖ in Example 2.4.2 for three values of σ.

τ	ϖ		
	$\boldsymbol{\sigma = \frac{4}{3}}$	$\boldsymbol{\sigma = \frac{5}{2}}$	$\boldsymbol{\sigma = \frac{7}{4}}$
0.00	0.0000	0.0000	0.0000
0.05	0.3110	0.1695	0.0664
0.10	0.3940	0.2414	0.1128
0.15	0.4533	0.2978	0.1544
0.20	0.5015	0.3464	0.1935
0.25	0.5430	0.3901	0.2310
0.30	0.5800	0.4303	0.2674
0.35	0.6137	0.4681	0.3030
0.40	0.6449	0.5040	0.3381
0.45	0.6742	0.5383	0.3728
0.50	0.7018	0.5713	0.4073
0.55	0.7281	0.6033	0.4415
0.60	0.7532	0.6345	0.4756
0.65	0.7774	0.6649	0.5095
0.70	0.8008	0.6946	0.5435
0.75	0.8234	0.7239	0.5775
0.80	0.8455	0.7526	0.6114
0.85	0.8669	0.7809	0.6455
0.90	0.8879	0.8088	0.6796
0.95	0.9084	0.8364	0.7138
1.00	0.9285	0.8637	0.7481

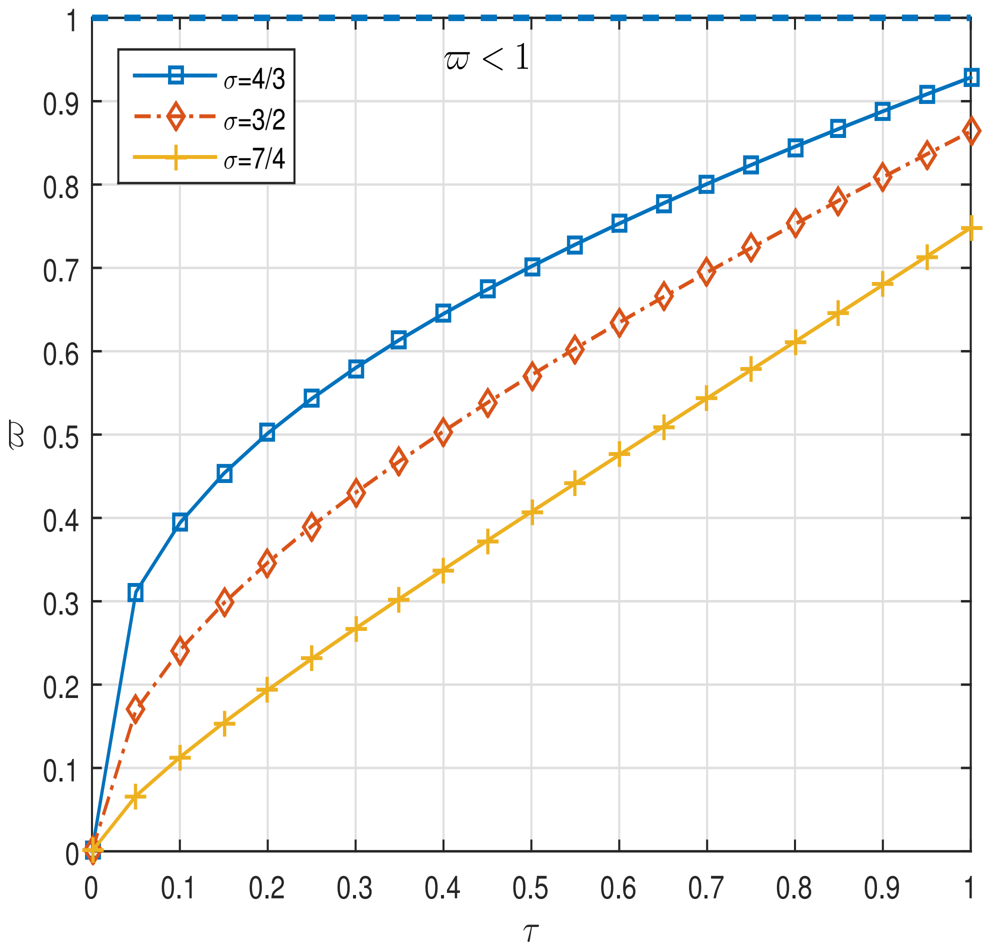

Figure 2.2: Representation of ϖ for BVP (2.38) in Example 2.4.2 for three case σ.

CHAPTER 3

EXISTENCE OF SOLUTIONS FOR NONLINEAR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Introduction

In this chapter, we study the existence of nontrivial solution for the fractional differential equation of order α with three-point boundary conditions having the following form

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\nu}v(t)), & t \in (0, T) \\ u(0) = 0, \quad u(T) = au(\xi), \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$, $\nu, a > 0$, $\xi \in (0, T)$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$. D is the standard Riemann-Liouville fractional derivative operator and $f \in C([0, 1] \times \mathbf{R}^2, \mathbf{R})$. Applying the Leray-Schauder nonlinear alternative, we prove the existence of at least one solution. As an application, we have also given some examples to illustrate the results obtained.

3.2 Green Function Of The Fractional Differential Equation

This section presents a lemma that will be used to determine the problem's Green function, thus making the analysis of problem (3.1) more tractable.

Lemma 3.2.1 *Let $y \in C([0, T])$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$. Then FDE*

$$\begin{cases} D^{\alpha}u(t) = y(t), & t \in (0, T) \\ u(0) = 0, \quad u(T) = au(\xi) \end{cases}$$

has a unique solution

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})} \right] y(s)ds$$
$$-\frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)} \int_t^T (t(T-s))^{\alpha-1} y(s)ds$$
$$+\frac{a}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (t(\xi-s))^{\alpha-1} y(s)ds$$

Proof 3.2.1 *([1])The general solution of FDE is*

$$u(t) = I^\alpha y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \text{ where } \ c_1, c_2 \in \mathbf{R}.$$

Using the boundary conditions, we find that $c_2 = 0$ *and*

$$c_1 = -\frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})} \left[\int_0^T \frac{y(s)ds}{(T-s)^{\alpha-1}\Gamma(\alpha)} - a\int_0^\xi \frac{y(s)ds}{(\xi-s)^{\alpha-1}\Gamma(\alpha)} \right]$$

Substituting c_1 *and* c_2 *by their values in* $u(t)$*, we obtain the solution in the statement of the lemma. This completes the proof.*

Define the integral operator $F : E \longrightarrow E$*, by*

$$Fu(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})} \right] f(s, v(s), D^\nu v(s))ds$$
$$-\frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)} \int_t^T (t(T-s))^{\alpha-1} f(s, v(s), D^\nu v(s))ds$$
$$+\frac{a}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (t(\xi-s))^{\alpha-1} f(s, v(s), D^\nu v(s))ds$$

By Lemma 3.2.1, the FDE (3.1) has a solution if and only if the operator F *has a fixed point in* E*. So we only need to seek a fixed point of* F *in* E*. By the Ascoli-Arzela theorem, we can prove that* F *is a completely continuous operator. According to this, we cited the Leray-Schauder nonlinear alternative* **??***.*

3.3 Existence Of Nontrivial Solution

In this section, we prove the existence of a nontrivial solution for the FDE (3.1)*. Let* $E = C([0, T])$ *with the norm* $\|v\| = \max_{t\in[0,T]}\{|v(t)|, |D^\nu v(t)|\}$ *for any* $v \in E$*,* $f \in C([0, T] \times \mathbf{R}^2, \mathbf{R})$*.*

Theorem 3.3.1 *Suppose that* $f(t, 0, 0) \neq 0$*,* $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$*, and there exist nonnegative functions* $k, h, l \in L^1[0, T]$ *such that*

$$|f(t, x, y)| \leq k(t)|x| + h(t)|y| + l(t), \quad a.e. \ \ (t, x, y) \in [0, T] \times \mathbf{R}^2,$$
$$\frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s) + h(s))ds$$
$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}(k(s) + h(s))ds < 1.$$

Then the FDE (3.1) has at least one nontrivial solution $u^* \in C([0, T])$*.*

Proof 3.3.1 *Let*

$$A = \frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s)+h(s))ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))ds,$$

$$B = \frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}l(s)ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}l(s)ds.$$

Then $A < 1$*. Since* $f(t,0,0) \neq 0$*, there exists an interval* $[a,b] \subset [0,1]$ *such that* $\min_{a \le t \le b} |f(t,0,0)| > 0$*. And as* $l(t) \ge |f(t,0,0)|$*, a.e.* $t \in [0,T]$*, we know that* $B > 0$*.*

Let $C = B(1-A)^{-1}$ *and* $\Omega = \{(u,v) \in E^2 : \|(u,v)\|_{E^2} < C\}$*. Assume that* $u \in \partial\Omega$ *and* $\lambda > 1$ *such that* $Fu = \lambda u$*, then*

$$\lambda C = \lambda\|u\| = \|Fu\| = \max_{0 \le t \le T} |(Fu)(t)|$$

$$\le \frac{1}{\Gamma(\alpha)} \max_{t\in[0,T]} \int_0^t \left|(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})}\right| |f(s,v(s),D^\nu v(s))|ds$$

$$+ \max_{t\in[0,T]} \frac{1}{|T^{\alpha-1} - a\xi^{\alpha-1}|\Gamma(\alpha)} \int_t^T (t(T-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

$$+ \max_{t\in[0,T]} \frac{a}{|T^{\alpha-1} - a\xi^{\alpha-1}|\Gamma(\alpha)} \int_0^\xi (t(\xi-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^T \left[(T-s)^{\alpha-1} + \frac{(T(T-s))^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})}\right] |f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{a}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (T(\xi-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

$$\le \frac{(2T^{\alpha-1} + a\xi^{\alpha-1})}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s)|v(s)| + h(s)|D^\nu v(s)| + l(s))ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}(k(s)|v(s)| + h(s)|D^\nu v(s)| + l(s))ds$$

$$\le \Bigg[\frac{(2T^{\alpha-1} + a\xi^{\alpha-1})}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s)+h(s))\|v\|ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))\|v\|ds\Bigg]$$

$$+\Bigg[\frac{(2T^{\alpha-1} + a\xi^{\alpha-1})}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}l(s)ds$$

$$
\begin{aligned}
&+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}l(s)ds\Big]\\
&=A\|v\|+B.
\end{aligned}
$$

Therefore,

$$
\lambda\leq A+\frac{B}{C}=A+\frac{B}{B(1-A)^{-1}}=A+(1-A)=1.
$$

This contradicts $\lambda>1$. By Lemma 1.5.2 , F has a fixed point $u^*\in\overline{\Omega}$. In view of $f(t,0,0)\neq 0$ the FDE (3.1) has a nontrivial solution $u^*\in E$.

Now, we prove that the operator F is completely continuous, we have $\mathrm{B}_C=\{v\in E:||v||\leq C\}$ is a bounded closed convex set of E. We shall prove that $F(\mathrm{B}_C)$ is relatively compact. The proof will be done is some steps.

(i) Let $v\in\mathrm{B}_C$, we have

$$
|Fu(t)|\leq A\|v\|+B.
$$

Consequently $F(\mathrm{B}_C)$ is uniformly bounded.

(ii) Let us prove that $F(\mathrm{B}_C)$ is equicontinuous. Let $t_1,t_2\in[0,T],t_1<t_2,v\in\mathrm{B}_C$. We have

$$
\begin{aligned}
\Big|Fu(t_1)-Fu(t_2)\Big| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-\frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right]f(s,v(s),D^{\nu}v(s))ds\\
&-\frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{T}(t_1(T-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds\\
&+\frac{a}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t_1(\xi-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1}-\frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right]f(s,v(s),D^{\nu}v(s))ds\\
&+\frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^{T}(t_2(T-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds\\
&-\frac{a}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t_2(\xi-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds\Big|\\
&=\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-\frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right]f(s,v(s),D^{\nu}v(s))ds\\
&-\frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{T}(t_1(T-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1}-\frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right]f(s,v(s),D^{\nu}v(s))ds\\
&+\frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^{T}(t_2(T-s))^{\alpha-1}f(s,v(s),D^{\nu}v(s))ds
\end{aligned}
$$

$$+\frac{a(t_1^{\alpha-1}-t_2^{\alpha-1})}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}f(s,v(s),D^\nu v(s))ds\Big|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}+\frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right]|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^T(t_1(T-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1}+\frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right]|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^T(t_2(T-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{a|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

Therefore,

$$\Big|Fu(t_1)-Fu(t_2)\Big|\leq\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\Big[((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})$$

$$+\frac{[(t_1(T-s))^{\alpha-1}-(t_2(T-s))^{\alpha-1}]}{(T^{\alpha-1}+a\xi^{\alpha-1})}\Bigg]|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{t_2}[(t_1(T-s))^{\alpha-1}-(t_2(T-s))^{\alpha-1}]|f(s,v(s),D^\nu v(s))|ds$$

$$+\frac{a|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds$$

Letting $t_1\to t_2$, then $|Fu(t_1)-Fu(t_2)|$ tends to 0. Consequently $F(\mathrm{B}_C)$ is equicontinuous. From Ascoli-Arzela theorem, we deduce that F is a completely continuous.

This completes the proof.

Theorem 3.3.2 *Suppose that $f(t,0,0)\neq 0$, $T^{\alpha-1}+a\xi^{\alpha-1}\neq 0$, and there exist nonnegative functions $k,h,l\in L^1[0,T]$ such that*

$$|f(t,x,y)|\leq k(t)|x|+h(t)|y|+l(t),\quad a.e.\ \ (t,x,y)\in[0,T]\times\mathbf{R}^2.$$

If one of the following conditions is fulfilled
(1) *There exists a constant $p>1$ such that*

$$\int_0^1(k(s)+h(s))^p ds<$$

$$\left[\frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}+aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p,\ \ \frac{1}{p}+\frac{1}{q}=1.$$

(2) $k(s)+h(s)$ *satisfies*

$$k(s)+h(s)\leq\frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^\alpha+aT^{\alpha-1}\xi^\alpha},\quad a.e.\quad s\in[0,T],$$

$$meas\left\{s\in[0,T]:k(s)+h(s)<\frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}\right\}>0.$$

Then the FDE (3.1) has at least one nontrivial solution $u^*\in E$.

Proof 3.3.2 *Let A be defined as in the proof of Theorem 3.3.1. To prove Theorem 3.3.2, we only need to prove that $A<1$. Since $T^{\alpha-1}+a\xi^{\alpha-1}\neq 0$, we have*

$$A=\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}(k(s)+h(s))ds.$$

(1) *Using the Hölder inequality, we have*

$$A\leq\left[\int_0^1(k(s)+h(s))^p ds\right]^{1/p}\left\{\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\left[\int_0^T\left((T-s)^{\alpha-1}\right)^q ds\right]^{1/q}\right.$$

$$\left.+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\left[\int_0^{\xi}\left((\xi-s)^{\alpha-1}\right)^q ds\right]^{1/q}\right\}$$

$$\leq\left[\int_0^1(k(s)+h(s))^p ds\right]^{1/p}\left\{\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\left[\frac{T^{1+q(\alpha-1)}}{(1+q(\alpha-1))}\right]^{1/q}\right.$$

$$\left.+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\left[\frac{\xi^{1+q(\alpha-1)}}{1+q(\alpha-1)}\right]^{1/q}\right\}$$

$$\leq\left[\int_0^1(k(s)+h(s))^p ds\right]^{1/p}\left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}\right.$$

$$\left.+\frac{aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}\right]$$

$$\leq\frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}+aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}$$

$$\times\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}+aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}=1.$$

(2) *In this case, we have*

$$A\leq\frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}\left[\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}ds\right.$$

$$\left.+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}ds\right]$$

$$\leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}\left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}\right.$$

$$\left.+\frac{aT^{\alpha-1}\xi^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}\right]$$

$$\leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}.\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}=1.$$

.

This completes the proof.

Corollary 3.3.1 *Suppose $f(t,0,0)\neq 0$, $(1+a)T^{\alpha-1}\neq 0$, and there exist nonnegative functions $k,h,l\in L^1[0,T]$ such that*

$$|f(t,x,y)|\leq k(t)|x|+h(t)|y|+l(t),\quad a.e.\ \ (t,x,y)\in[0,T]\times\mathbf{R}^2.$$

If one of the following conditions is fulfilled
(1) *There exists a constant $p>1$ such that*

$$\int_0^1(k(s)+h(s))^p ds<\left[\frac{(1+a)T^{\alpha-1}\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{2(1+a)T^{\alpha-1}T^{(1+q(\alpha-1))/q}}\right]^p,\ \frac{1}{p}+\frac{1}{q}=1.$$

(2) *$k(s)+h(s)$ satisfies*

$$k(s)+h(s)\leq\frac{\alpha\Gamma(\alpha)}{2T^{\alpha}},\quad a.e.\quad s\in[0,T],$$

$$meas\left\{s\in[0,T]:k(s)+h(s)<\frac{\alpha\Gamma(\alpha)}{2T^{\alpha}}\right\}>0.$$

Then the FDE (3.1) has at least one nontrivial solution $u^*\in E$.

Proof 3.3.3 *In this case, we have*

$$A=\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}(k(s)+h(s))ds$$

$$\leq\frac{2T^{\alpha-1}+aT^{\alpha-1}}{(T^{\alpha-1}+aT^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds$$

$$+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+aT^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds$$

$$=\frac{2(1+a)T^{\alpha-1}}{(1+a)T^{\alpha-1}\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds.$$

Proof of this corollary 3.3.1 is the same method as in the proof of the theorem 3.3.2. This completes the proof.

3.4 Examples

In order to illustrate the above results, we consider some examples.

Example 3.4.1 *Consider the following system of FDE*

$$\begin{cases} D^{3/2}u(t) = \frac{t}{207}v(t) + \frac{t+2}{100}D^{5/4}v(t) + t^2 - 1, \quad t \in (0,T) \\ \\ u(0) = 0, \quad u(T) = 2u(T/2). \end{cases} \tag{3.2}$$

Set $\alpha = 3/2$, $a = 2$, $\xi = T/2$, *and*

$$f(t,x,y) = \frac{t}{207}x(t) + \frac{t+2}{100}y(t) + t^2 - 1,$$

$$k(t) = \frac{t}{100}, \quad h(t) = \frac{t+2}{100}, \quad l(t) = t^2.$$

It is easy to prove that $k, h, l \in L^1[0,T]$ *are nonnegative functions,*
and

$$|f(t,x,y)| \leq k(t)|x| + h(t)|y| + l(t), \quad a.e. \;\; (t,x,y) \in [0,T] \times \mathbf{R}^2,$$

and

$$T^{\alpha-1} + a\xi^{\alpha-1} = (1 + \frac{2}{2^{1/2}})T^{1/2} \neq 0.$$

Moreover, we have

$$A = \frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s) + h(s))ds$$

$$+ \frac{aT^{\alpha-1}}{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1}(k(s) + h(s))ds,$$

$$A \approx 13.10^{-3}.T^{3/2} + 4.10^{-3}.T^{5/2} < 1.$$

Hence, by Theorem 3.3.1, the FDE 3.2 has at least one nontrivial solution u^* *in* E.

Example 3.4.2 *Consider the following system of FDE*

$$\begin{cases} D^{1/2}u(t) = \frac{\sqrt[3]{1+t^5}}{20}v(t)\sin v(t) + \frac{\sqrt[3]{1+t^5}}{5}D^{3/4}v(t) + \cos t - e^t, \quad t \in (0,T) \\ \\ u(0) = 0, \quad u(T) = 4u(T/3). \end{cases} \tag{3.3}$$

Set $\alpha = 1/2$, $a = 4$, $\xi = T/3$, *and*

$$f(t,x,y) = \frac{\sqrt[3]{1+t^5}}{20}x(t)\sin x(t) + \frac{\sqrt[3]{1+t^5}}{5}y(t) + \cos t - e^t,$$

$$k(t) = \sqrt[3]{1+t^5}/10, \quad h(t) = \sqrt[3]{1+t^5}/4, \quad l(t) = \cos t + e^t.$$

It is easy to prove that $k, h, l \in L^1[0,T]$ *are nonnegative functions,*
and

$$|f(t,x,y)| \leq k(t)|x| + h(t)|y| + l(t), \quad a.e. \;\; (t,x,y) \in [0,T] \times \mathbf{R}^2.$$

and

$$T^{\alpha-1} + a\xi^{\alpha-1} = (1 + \frac{4}{3^{-1/2}})T^{-1/2} \neq 0.$$

Let $p = 3,\ q = 3/2,$ *such that* $\frac{1}{p} + \frac{1}{q} = 1,$ *then*

$$\int_0^1 (k(s) + h(s))^p ds = \frac{2401}{48000} \approx 0.05$$

Moreover, we have

$$\left[\frac{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)(1 + q(\alpha-1))^{1/q}}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q} + aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p \approx 0.51.T^{-1/2}$$

Therefore,

$$\int_0^1 (k(s) + h(s))^p ds <$$

$$\left[\frac{(T^{\alpha-1} + a\xi^{\alpha-1})\Gamma(\alpha)(1 + q(\alpha-1))^{1/q}}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q} + aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p.$$

Hence, by Theorem 3.3.2 (1)*, the FDE 3.3 has at least one nontrivial solution* u^* *in* E*.*

Example 3.4.3 *Consider the following system of FDE*

$$\begin{cases} D^{3/2}u(t) = \frac{\sqrt{t}}{2(\frac{1}{2}+v(t))}e^{|v^2(t)-1|}\cos v(t) + \frac{(1+t^2)}{9+e^t}D^{7/3}v(t) + e^{-t} - \sin t, \quad t \in (0,T) \\ \\ u(0) = 0, \quad u(T) = 3u(T/4). \end{cases} \tag{3.4}$$

Set $\alpha = 3/2,\ a = 3,\ \xi = T/4,$ *and*

$$f(t,x,y) = \frac{\sqrt{t}}{2(\frac{1}{2} + x(t))}e^{|x^2(t)-1|}\cos x(t) + \frac{(1+t^2)}{9+e^t}y(t) + e^{-t} - \sin t,$$

$$k(t) = \frac{\sqrt{t}}{2}, \quad h(t) = \frac{(1+t^2)}{3}, \quad l(t) = e^{-t} + \sin t.$$

It is easy to prove that $k, h, l \in L^1[0,T]$ *are nonnegative functions, and*

$$|f(t,x,y)| \leq k(t)|x| + h(t)|y| + l(t), \quad a.e. \ \ (t,x) \in [0,T] \times \mathbf{R}^2.$$

and

$$T^{\alpha-1} + a\xi^{\alpha-1} = (1 + \frac{3}{4^{1/2}})T^{1/2} \neq 0.$$

Moreover, we have

$$\frac{\alpha\Gamma(\alpha)(T^{\alpha-1} + a\xi^{\alpha-1})}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^\alpha + aT^{\alpha-1}\xi^\alpha} = \frac{15\sqrt{\pi}}{31}T^{-3/2}.$$

Therefore,

$$k(s) + h(s) = \frac{\sqrt{s}}{2} + \frac{(1+s^2)}{3} < \frac{15\sqrt{\pi}}{31}T^{-3/2}, \quad s \in [0,T],$$

$$meas\{s \in [0,T] : k(s) + h(s) < \frac{\alpha\Gamma(\alpha)(T^{\alpha-1} + a\xi^{\alpha-1})}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^\alpha + aT^{\alpha-1}\xi^\alpha}\} > 0.$$

Hence, by Theorem 3.3.2 (2)*, the FDE 3.4 has at least one nontrivial solution* u^* *in* E*.*

CONCLUSION

In this dissertation, , we have investigated two classes of fractional boundary value problems (BVPs) involving distinct fractional operators: the Caputo–Katugampola derivative and the classical Riemann–Liouville derivative.

*In **Chapter 2**, we studied a nonlinear BVP of order* $1 < \sigma \leq 2$ *with two-point boundary conditions. By constructing the associated Green's function and applying the Banach contraction principle, we established sufficient conditions for the existence and uniqueness of a solution. Numerical simulations were provided to illustrate the theoretical results and to show the influence of the fractional order* σ *on the solution behavior.*

*In **Chapter 3**, we considered a fractional differential equation of order* $\alpha \in (1, 2)$ *with three-point boundary conditions. Using the Leray–Schauder nonlinear alternative and properties of the Green function, we proved the existence of at least one nontrivial solution under suitable growth conditions on the nonlinear term. Several examples were presented to validate the obtained criteria.*

The main contributions of this work are:

- *Extension of classical boundary value problems to the fractional setting.*
- *Derivation of explicit Green's functions and precise estimates for integral kernels.*
- *Application of fixed-point theorems (Banach, Leray–Schauder) in the context of fractional derivatives.*
- *Numerical verification of theoretical conditions using different fractional orders and boundary values.*

***Perspectives for future work** include:*

- *Studying coupled systems of fractional BVPs.*
- *Investigating the same problems with nonlocal or integral boundary conditions.*
- *Developing numerical schemes (e.g., finite difference, spline collocation) to approximate solutions.*
- *Extending the analysis to fractional derivatives with variable order or singular kernels.*

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Abstract

This work studies two nonlinear fractional boundary value problems involving the Caputo-Katugampola and Riemann-Liouville derivatives; using theorems (Banach Contraction, leray-schauder nonlinear alternative), it establishes existence, uniqueness and nontrivial existence results for the solution.

Résumé

Ce travail étudie deux problèmes aux limites fractionnaires non linéaires faisant intervenir les dérivées de Caputo-Katugampola et de Riemann-Liouville. En s'appuyant sur des théorèmes tels que le principe de contraction de Banach et l'alternative non linéaire de Leray-Schauder, il établit des résultats concernant l'existence, l'unicité et l'existence de solutions non triviales.

ملخص

تتناول هذه الدراسة مسألتين غير خطيتين من مسائل القيم الحدية الكسرية تتضمنان مشتقات كابوتو-كاتوغامبولا وريمان-ليوفيل.

باستخدام نظريات (انكماش باناخ، وبديل ليري-شودر غير الخطي)، تُثبت الدراسة وجود الحلول ووحدانيتها بالإضافة بالإضافة إلى نتائج وجود حلول غير بديهية.