University of Belhadj Bouchaib - Ain Temouchent
Faculty of Science and Technology

Department of Mathematics and Computer Science
Engineering and Sustainable Development Laboratory

Teaching Handout

Kheira MEKHALFI

Title : ALGEBRA 4

Courses and Corrected Exercises for Students of :
2nd-Year of a Bachelor's degree in Mathematics

Academic Year : 2025-2026

Foreword

𝔗his handout is the support material for the Algebra 4 course teaching at Belhadj Bouchaib University in Ain Témouchent for second-year mathematics undergraduates by Kheira Mekhalfi. It was transcribed throughout the year and cannot in any way replace the actual course.

This document closely follows the course as it was taught, and apart from a few minor modifications, it reproduces the lectures as they were delivered to all students.

For any remarks, suggestions, or corrections regarding this document, please contact me so that I can update and improve this handout.

Author: K. Mekhalfi

INTRODUCTION

The notion of bilinear form is defined on vector spaces, which are special cases of bilinear applications on a Cartesian product of two vector spaces in a vector space (where all the spaces involved are defined on the same body). These forms are closely linked to linear applications. The knowledge associated with the latter makes it possible to shed light on the structure of a bilinear form. Some bilinear forms are also scalar products. Scalar products (on finite or infinite dimensional vector spaces) are widely used, in all branches of mathematics, to define a distance.

Classical, relativistic or quantum physics uses this formal framework. Geometry uses the scalar product to define distance, orthogonality, angle, ... Number theory uses quadratic forms to demonstrate or solve certain purely algebraic problems. Sometimes, linking mathematical branches, such as number theory and algebraic geometry, such as the search for solutions to a Diophantine equation. Some of them are written as the search for the roots of a polynomial equation with several variables and integer coefficients. The solutions sought are those that are expressed only with integers. A famous and difficult example is Fermat's great theorem. The equation is written $x^n + y^n = z^n$ (for $n = 2$, the solutions are the Pythagorean triplets, which are called Fermat's two-square theorem). The solutions can be seen as points of intersection between $\mathbb{Z}^3$ and a surface of a geometric space of dimension three. To be compatible with the ministerial program, we limit ourselves to bilinear forms on a finite-dimensional vector space (i.e. the Cartesian product of a vector space in itself), in particular, the quadratic forms taken are those of the symmetric bilinear forms.

It is composed of five chapters:

In Chapter 1: we recall some definitions and give without proof some classical results on vector spaces and linear mappings, that is, we list in this chapter the basic notions on a vector space and its dual space.

In chapter 2: we deal with bilinear forms over a real vector space. It is not possible to understand such properties without examining the related concepts of linear forms. More precisely, this chapter describes the most properties of bilinear forms on a vector space and gives examples of the three most common types of such forms as well as symmetric, skew-symmetric and alternating bilinear forms.

In chapter 3: we deal with the important condition of nondegeneracy for a bilinear form, Gauss decomposition theorem and the orthogonal basis for a symmetric bilinear form. In chapter 4: we recall an introduction to Hermitian space is given .

In chapter 5: we deals with the spectral decomposition of selfadjoint linear mappings.

At the end of this lecture-note, the reader will find a conclusion and a bibliography.

Chapter 1

Linear forms, Duality

Introduction

As a continuation of Algebra II, this chapter explores the relationships between scalars, vectors, and multivariable linear maps on finite-dimensional vector spaces. Recall that a vector space is a fundamental object in linear algebra: a set whose elements, called vectors, can be added together and scaled by numbers (scalars). Here we focus on linear mappings defined on particular vector spaces that involve one or two variables, and we examine how these maps connect scalars and vectors within a finite-dimensional setting see [1, 5].

1.1 Linear Forms

Throughout this chapter $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ will designate a commutative field and E a $\mathbb{K}-$vector space (finite-dimensional or not). Let $f : E \to F$ be a mapping from E to F.

Definition 1.1.1 *A linear form on a finite-dimensional $\mathbb{K}$-vector space E of dimension n is any linear map $\varphi : E \to \mathbb{K}$.*
A linear form is thus a function $\varphi : E \to \mathbb{K}$ satisfying

$$\forall x, y \in E,\ \forall \lambda \in \mathbb{K} : \quad \varphi(\lambda x + y) = \lambda \varphi(x) + \varphi(y).$$

Example 1.1.1 *1. Let $E = \mathcal{C}([0,1], \mathbb{K})$ be the vector space of continuous functions on $[0,1]$.*

The map

$$I : E \to \mathbb{K}, \qquad I(f) = \int_0^1 f(t)\,dt$$

is a linear form on E*.*

2. *Let* $\mathcal{M}_n(\mathbb{K})$ *be the vector space of* $n \times n$ *matrices with entries in* $\mathbb{K}$*. The trace map*

$$\mathrm{Tr} : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}, \qquad \mathrm{Tr}(M) = \sum_{i=1}^{n} M_{ii}$$

is a linear form on $\mathcal{M}_n(\mathbb{K})$*.*

3. *If* E *has dimension* n *and* $B = (e_j)_{1 \leq j \leq n}$ *is a basis of* E*, the coordinate projections relative to* B

$$p_j : E \to \mathbb{K}, \qquad x = \sum_{i=1}^{n} x_i e_i \longmapsto x_j$$

are linear forms on E*.*

4. *The zero map that assigns the scalar* $0 \in \mathbb{K}$ *to every vector of* E *is a linear form on* E*.*

5. *Let* $a_1, a_2 \in \mathbb{R}$*. The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x_1, x_2) = a_1 x_1 + a_2 x_2$ *is a linear form on* $\mathbb{R}^2$*.*

1.2 Hyperplans

Proposition 1.2.1 *Let* f *be a non-zero linear form on a vector space* E *of dimension* n*. Then:*

$$\dim(\ker f) = n - 1$$

Proof. According to the rank theorem, we have:

$$\dim(\ker f) + \dim(\operatorname{im} f) = \dim E$$

Since f is non-zero, $\operatorname{im} f \neq \{0_{\mathbb{K}}\}$ and $\operatorname{im} f \subseteq \mathbb{K}$, then:

$$0 < \dim(\operatorname{im} f) \leq \dim \mathbb{K} = 1$$

Thus, $\dim(\operatorname{im} f) = 1$, and therefore:

$$\dim(\ker f) = n - 1$$

■

Remark 1.2.1 *Proposition (1.2.1) leads to a generalization of the notion of a hyperplane to vector spaces of finite or infinite dimension.*

Definition 1.2.1 *A hyperplane of E is a vector subspace $H \subset E$ such that there exists a non-zero linear form $f : E \to \mathbb{K}$ with $H = \ker f$.*

Remark 1.2.2 *If the dimension of E is finite and equal to n, then by Proposition (1.2.1), we recover the classical definition of a hyperplane H of E; H is a vector subspace of E of dimension* $n - 1$.

Example 1.2.1

1. *The set*

$$H = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\}$$

   is a hyperplane of $\mathbb{R}^3$, because if $f : \mathbb{R}^3 \to \mathbb{R}$ is the map defined by

$$f(x, y, z) = x + 2y - z,$$

   then f is a non-zero linear form and $\ker f = H$.

2. *The set*

$$H = \{f \in \mathcal{C}([0, 1], \mathbb{R}) : f(0) = 0\}$$

   is a hyperplane of $\mathcal{C}([0, 1], \mathbb{R})$, because if $g : \mathcal{C}([0, 1], \mathbb{R}) \to \mathbb{R}$ is the map defined by

$$g(f) = f(0),$$

   then g is a non-zero linear form and $\ker g = H$.

Proposition 1.2.2 *Let H be a vector subspace of E. The following two properties are equivalent:*

1. *H is a hyperplane of E.*
2. *There exists a one-dimensional subspace $D \subset E$ such that $E = H \oplus D$.*

Moreover, if E is finite-dimensional with $\dim(E) = n$, then the two previous properties are equivalent to:

3. $\dim(H) = \dim(E) - 1$ *(i.e., H has codimension 1).*

Proof.

Let H be a hyperplane of E, so there exists a non-zero linear form $\varphi \in E^* \setminus \{0\}$ such that $H = \ker(\varphi)$. Since φ is not identically zero, there exists $v \in E$ such that $\varphi(v) \neq 0$. Let $D = \text{span}(v)$, and we show that $E = H \oplus D$. Let $x \in H \cap D$. Then $\varphi(x) = 0$ and $x = \lambda v$ for some $\lambda \in \mathbb{K}$, so:

$$\varphi(x) = \varphi(\lambda v) = \lambda \varphi(v)$$

Since $\varphi(v) \neq 0$, it follows that $\lambda = 0$, hence $x = 0$. Therefore, $H \cap D = \{0\}$.
Now let $x \in E$. Define:

$$\lambda = \frac{\varphi(x)}{\varphi(v)}, \quad \text{and} \quad y = x - \lambda v.$$

Then:

$$\varphi(y) = \varphi(x - \lambda v) = \varphi(x) - \lambda \varphi(v) = 0.$$

So $y \in H$, and thus:

$$x = y + \lambda v \in H + D$$

Hence, $E = H \oplus D$. ■

Remark 1.2.3 *If* $\dim(E) = n$ *is finite, and* $\mathcal{B}_E = \{e_1, e_2, \dots, e_n\}$ *is a basis of* E*, then relative to the basis* $\mathcal{B}_E$*, a hyperplane* $H \subset E$ *admits a unique equation, up to a scalar multiple, of the form:*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0, \quad \textit{where } x = (x_1, x_2, \dots, x_n)_{\mathcal{B}_E} \in E$$

Corollaire 1.2.1 *Two non-zero linear forms on a vector space E are proportional if and only if they have the same kernel.*

Proof. Let φ and ψ be two non-zero linear forms. Suppose $\ker(\psi) = \ker(\varphi) = H$. Let $v \notin H$, which implies $\psi(v) \neq 0$ and $\varphi(v) \neq 0$. Define:

$$\lambda = \frac{\varphi(v)}{\psi(v)}$$

Let $x \in E$. By Proposition (1.2.2), we have $E = H \oplus \text{span}(v)$, so there exist $y \in H$ and $\mu \in \mathbb{K}$ such that:

$$x = y + \mu v$$

Then:

$$\varphi(x) = \varphi(y + \mu v) = \varphi(y) + \mu\varphi(v) = \mu\varphi(v)$$

Since $\varphi(v) = \lambda\psi(v)$, we get:

$$\varphi(x) = \mu\lambda\psi(v) = \lambda(\psi(y) + \mu\psi(v)) = \lambda\psi(x)$$

Conclusion: For all $x \in E$, there exists $\lambda \in \mathbb{K}$ such that $\varphi(x) = \lambda\psi(x)$, which shows that φ and ψ are proportional.

The reverse implication is immediate. ■

1.3 Duality

1.3.1 Dual Vector Space

Definition 1.3.1 *The dual vector space of E, denoted E^*, is the vector space of all linear forms on E:*

$$E^* = \mathcal{L}(E, \mathbb{K})$$

Proposition 1.3.1 *If E is finite-dimensional, then E^* is also finite-dimensional, and we have:*

$$\dim E = \dim E^*$$

Proof. If E is finite-dimensional, then $\mathcal{L}(E, \mathbb{K})$ is also finite-dimensional, and:

$$\dim E^* = \dim \mathcal{L}(E, \mathbb{K}) = \dim E \times \dim \mathbb{K} = \dim E.$$

Since: $\mathbb{K}$ is the base field and $\dim \mathbb{K} = 1$. ■

1.3.2 Dual Basis

Let us recall that a basis of a vector space E is a family of vectors such that every vector in E can be written uniquely as a linear combination of that family.

A linear map is therefore uniquely determined by specifying the image of each basis vector of the domain space.

Let E be a finite-dimensional vector space. If $B = \{e_1, \ldots, e_n\}$ is a basis of E, then we can associate to it a basis $B^* = \{e_1^*, \ldots, e_n^*\}$ of E^*, called the **dual basis**.

Proposition 1.3.2 *Let $B = \{e_1, \ldots, e_n\}$ be a basis of a vector space E. For each $j \in \{1, \ldots, n\}$, there exists a unique linear form e_j^* such that:*

$$\forall i \in \{1, \ldots, n\} : e_j^*(e_i) = \delta_{ij} = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j \end{cases} \tag{1.1}$$

The form e_j^* is the j-th coordinate form with respect to the basis B, i.e.,

$$e_j^*(x_1 e_1 + \cdots + x_n e_n) = x_j.$$

Proof. A linear form from E to $\mathbb{K}$ is entirely determined by the image of each basis vector of E. Thus, for each fixed $j \in \{1, \ldots, n\}$, the n equations (1.1) uniquely define the linear form e_j^*. For all $(x_1, \ldots, x_n) \in \mathbb{K}^n$, we have:

$$\begin{aligned} e_j^*(e_i) &= e_j^*(x_1 e_1 + x_2 e_2 + \cdots + x_j e_j + \cdots + x_n e_n) \\ &= x_1 e_j^*(e_1) + x_2 e_j^*(e_2) + \cdots + x_j e_j^*(e_j) + \cdots + x_n e_j^*(e_n) \\ &= x_1 \cdot 0 + x_2 \cdot 0 + \cdots + x_j \cdot 1 + \cdots + x_n \cdot 0 \\ &= x_j. \end{aligned}$$

■

Theorem 1.3.1 *For every basis $B = \{e_1, \ldots, e_n\}$ of a vector space E, there exists a unique basis $B^* = \{e_1^*, \ldots, e_n^*\}$ of the dual space E^*, called the **dual basis** of B, such that for all $i, j \in \{1, \ldots, n\}$:*

$$e_j^*(e_i) = \delta_{ij}.$$

Proof. Existence and uniqueness are guaranteed by the previous proposition. To show that $\{e_1^*, \ldots, e_n^*\}$ is a basis of the dual space E^*, we must prove that it is both generating and linearly independent.

Generating: Let $\varphi \in E^*$, and let $x \in E$. Then:

$$x = \sum_{i=1}^{n} x_i e_i.$$

By linearity:

$$\varphi(x) = \varphi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \varphi(e_i) = \sum_{i=1}^{n} \varphi(e_i) e_i^*(x)$$

This shows that φ is a linear combination of the forms e_i^*, for $i = 1, \dots, n$.

Linearly independent: Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ such that:

$$\alpha_1 e_1^* + \cdots + \alpha_n e_n^* = 0.$$

Then for all $i \in \{1, \dots, n\}$:

$$0 = (\alpha_1 e_1^* + \cdots + \alpha_n e_n^*)(e_i) = \alpha_i.$$

Hence, all $\alpha_i = 0$, proving that the family $\{e_1^*, \dots, e_n^*\}$ is linearly independent. ■

Example 1.3.1 *Let $\{e_1, e_2, e_3\}$ be a basis of $\mathbb{R}^3$, where:*

$$e_1 = (1, 1, 1), \quad e_2 = (1, 0, -1), \quad e_3 = (0, 1, 1).$$

We want to determine the dual basis $\{e_1^*, e_2^*, e_3^*\}$ of $\mathbb{R}^{3*}$ corresponding to $\{e_1, e_2, e_3\}$.

We require:

$$e_1^*(e_1) = 1, \quad e_1^*(e_2) = 0, \quad e_1^*(e_3) = 0.$$

Let $e_1^*(x) = ax_1 + bx_2 + cx_3$, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then:

$$\begin{cases} a + b + c = 1 \\ a - c = 0 \\ b + c = 0 \end{cases} \Rightarrow a = 1, \quad b = -1, \quad c = 1.$$

So:

$$e_1^*(x) = x_1 - x_2 + x_3.$$

Similarly, we compute:

$$e_2^*(x) = x_2 - x_3, \quad e_3^*(x) = -x_1 + 2x_2 - x_3.$$

Thus, the dual basis of $\{e_1, e_2, e_3\}$ is:

$$\begin{aligned} e_1^*(x) &= x_1 - x_2 + x_3 \\ e_2^*(x) &= x_2 - x_3 \\ e_3^*(x) &= -x_1 + 2x_2 - x_3. \end{aligned}$$

Proposition 1.3.3 *Every basis of E^* is the dual basis of a unique basis of E, called the* ***pre-dual basis****.*

Proposition 1.3.4 (Dual Change of Basis) *Let B_1 and B_2 be two different bases of a vector space E, and let P be the change-of-basis matrix from B_1 to B_2. Then the change-of-basis matrix from the dual basis B_1^* to the dual basis B_2^* is $(P^{-1})^t$.*

Proof. Let:

- $B_1 = \{e_1, e_2, \dots, e_n\}$,
- $B_2 = \{f_1, f_2, \dots, f_n\}$,
- $P = (a_{ij})$ such that $f_k = \sum_{l=1}^n a_{lk} e_l$,
- $Q = (b_{ij})$ such that $f_j^* = \sum_{i=1}^n b_{ij} e_i^*$.

We compute:

$$\begin{aligned}
\delta_{jk} = f_j^*(f_k) &= \left(\sum_{i=1}^n b_{ij} e_i^*\right)\left(\sum_{l=1}^n a_{lk} e_l\right) \\
&= \sum_{i=1}^n \sum_{l=1}^n b_{ij} a_{lk} e_i^*(e_l) \\
&= \sum_{i=1}^n \sum_{l=1}^n b_{ij} a_{lk} \delta_{il} \\
&= \sum_{i=1}^n b_{ij} a_{ik}.
\end{aligned}$$

This is the (j,k)-th entry of the matrix Q^tP. Since $\delta_{jk} = f_j^*(f_k)$, we conclude:

$$Q^tP = I_n \quad \Rightarrow \quad Q = (P^{-1})^t.$$

■

Corollaire 1.3.1 (Practical Computation of the Dual Basis) *Let $\mathcal{B}_0 = \{e_1, e_2, \dots, e_n\}$ be the canonical basis of a vector space E, and let $\mathcal{B}_0^* = \{e_1^*, e_2^*, \dots, e_n^*\}$ be its dual basis. Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be another basis of E, and let $\mathcal{B}^* = \{v_1^*, v_2^*, \dots, v_n^*\}$ be its dual basis. Assume the vectors v_i (respectively v_i^*) are expressed in the canonical basis $\mathcal{B}_0$ (respectively $\mathcal{B}_0^*$). Then:*

$$Mat_{\mathcal{B}_0^*}(\mathcal{B}^*) = \left((Mat_{\mathcal{B}_0}(\mathcal{B}))^{-1}\right)^t.$$

That is, the matrix of the dual basis in the dual canonical basis is the transpose of the inverse of the matrix of the original basis in the canonical basis.

Example 1.3.2 (Computing the Dual Basis Matrix) *Let:*

$$v_1 = (3, 1, 1), \quad v_2 = (5, 2, 1), \quad v_3 = (6, 2, 1)$$

be vectors in $\mathbb{R}^3$ *expressed in the canonical basis* $\mathcal{B}_0 = \{e_1, e_2, e_3\}$*. Then the matrix of the basis* $\mathcal{B} = \{v_1, v_2, v_3\}$ *is:*

$$P = \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

The matrix of the dual basis $\mathcal{B}^* = \{\varphi_1, \varphi_2, \varphi_3\}$ *in the dual canonical basis* $\mathcal{B}_0^* = \{e_1^*, e_2^*, e_3^*\}$ *is:*

$$Q = (P^{-1})^t = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

This means:

$$\varphi_1 = 0 \cdot e_1^* + 1 \cdot e_2^* + 1 \cdot e_3^*.$$

$$\varphi_2 = 1 \cdot e_1^* + 3 \cdot e_2^* + 2 \cdot e_3^*.$$

$$\varphi_3 = 2 \cdot e_1^* + 0 \cdot e_2^* + 1 \cdot e_3^*.$$

1.4 Bidual of a Vector Space

Definition 1.4.1 *Let* E *be a vector space over a field* $\mathbb{K}$*. The dual of the dual space* E^**, denoted* E^{**}*, is called the* ***bidual*** *of* E*:*

$$E^{**} = (E^*)^*$$

Proposition 1.4.1 (Canonical Isomorphism with the Bidual) *Let* E *be a vector space over a field* $\mathbb{K}$*. Define the map:*

$$\Phi : E \to E^{**}, \quad x \mapsto \tilde{x}$$

where $\tilde{x} \in E^{**}$ *is defined by:*

$$\tilde{x}(\varphi) = \varphi(x), \quad \textit{for all } \varphi \in E^*$$

Then Φ *is a vector space isomorphism between* E *and its bidual* E^{**}*.*

Proof.

1. Linearity:

   Let $x, y \in E$, $\lambda \in \mathbb{K}$. Then:

   $$\widetilde{(x+\lambda y)}(\varphi) = \varphi(x+\lambda y) = \varphi(x) + \lambda\varphi(y) = \tilde{x}(\varphi) + \lambda\tilde{y}(\varphi).$$

   So:

   $$\widetilde{(x+\lambda y)} = \tilde{x} + \lambda\tilde{y}$$

   Hence, Φ is linear.

2. Injectivity:

   Suppose $\Phi(x) = 0 \in E^{**}$. Then:

   $$\varphi(x) = 0 \quad \text{for all } \varphi \in E^*.$$

   This implies $x \in \bigcap_{\varphi \in E^*} \ker(\varphi)$. In finite dimension, this intersection is $\{0\}$, so $x = 0$. Hence, Φ is injective.

3. Surjectivity:

   Since $\dim(E) = \dim(E^{**})$ and Φ is linear and injective, it is also surjective.

The map $\Phi : E \to E^{**}$, defined by $x \mapsto \tilde{x}$, is a vector space isomorphism. ■

1.5 Transpose and Annihilator

1.5.1 Annihilators and Orthogonals

Let E be a vector space over a field $\mathbb{K}$.

1. For any subset $F \subseteq E$, the **annihilator** of F is defined as:

   $$F^\perp = \{\varphi \in E^* \mid \forall x \in F,\ \varphi(x) = 0\}.$$

2. For any subset $G \subseteq E^*$, the **orthogonal** of G in E is:

   $$G^\perp = \{x \in E \mid \forall \varphi \in G,\ \varphi(x) = 0\}.$$

Theorem 1.5.1 *Let E be a finite-dimensional vector space over a field $\mathbb{K}$, and let $F \subseteq E$ be a subspace. Then the annihilator of F, defined as:*

$$F^{\perp} = \{\varphi \in E^{*} \mid \forall x \in F,\ \varphi(x) = 0\},$$

satisfies the dimension relation:

$$\dim(F) + \dim(F^{\perp}) = \dim(E)$$

Proprieties 1.5.1 *Let E be a finite-dimensional vector space over a field $\mathbb{K}$. Let $F \subseteq E$ and $G \subseteq E^{*}$ be subspaces.*

1. ***Annihilators are subspaces:*** *$F^{\perp} \subseteq E^{*}$ and $G^{\perp} \subseteq E$ are vector subspaces.*

2. ***Dimension formula:***

$$\dim(F) + \dim(F^{\perp}) = \dim(E), \quad \dim(G) + \dim(G^{\perp}) = \dim(E).$$

3. ***Double orthogonal (in finite dimension):***

$$(F^{\perp})^{\perp} = F, \quad (G^{\perp})^{\perp} = G.$$

4. ***Reversal of inclusion:***

$$F_1 \subseteq F_2 \Rightarrow F_2^{\perp} \subseteq F_1^{\perp}, \quad G_1 \subseteq G_2 \Rightarrow G_2^{\perp} \subseteq G_1^{\perp}.$$

5. ***Annihilator of sum and intersection:***

$$(F_1 + F_2)^{\perp} = F_1^{\perp} \cap F_2^{\perp}, \quad (F_1 \cap F_2)^{\perp} = F_1^{\perp} + F_2^{\perp}.$$

Example 1.5.1 *Let $E = \mathbb{R}^3$, and consider the subspace:*

$$F = \textit{span}\,\{(1,0,1),\ (0,1,1)\}$$

We want to compute its annihilator $F^{\perp} \subseteq E^{*}$.

Let $\varphi \in E^{*}$ be a linear form represented by $\varphi = (a,b,c) \in \mathbb{R}^3$. We require:

$$\varphi(1,0,1) = a + c = 0\varphi(0,1,1) = b + c = 0.$$

Solving:

$$a = -c, \quad b = -c \Rightarrow \varphi = c \cdot (-1, -1, 1).$$

So:

$$F^{\perp} = \mathit{span}\left\{(-1, -1, 1)\right\}, \quad \dim(F^{\perp}) = 1.$$

Since $\dim(F) = 2$, *we verify:*

$$\dim(F) + \dim(F^{\perp}) = 2 + 1 = 3 = \dim(E).$$

1.5.2 Transpose of a Linear Map

Definition 1.5.1 *Let* $f : E \to F$ *be a linear map between two vector spaces over a field* $\mathbb{K}$*. The* ***transpose*** *of* f*, denoted* $f^t : F^* \to E^*$*, is defined by:*

$$f^t(\varphi) = \varphi \circ f \quad \textit{for all } \varphi \in F^*.$$

That is, for every $x \in E$*:*

$$f^t(\varphi)(x) = \varphi(f(x))$$

Proprieties 1.5.2 *1.* ***Linearity:***

$$f^t(\lambda\varphi + \psi) = \lambda f^t(\varphi) + f^t(\psi) \quad \textit{for all } \varphi, \psi \in F^*,\ \lambda \in \mathbb{K}$$

2. ***Matrix Representation:*** *If* f *is represented by a matrix* A *in bases* $\mathcal{B}_E$ *and* $\mathcal{B}_F$*, then* f^t *is represented by the transpose matrix* A^T *in the dual bases* $\mathcal{B}_F^*$ *and* $\mathcal{B}_E^*$*.*

3. ***Kernel and Image Duality:***

$$\ker(f^t) = (\mathit{Im}(f))^{\perp}, \quad \mathit{Im}(f^t) = (\ker(f))^{\perp}$$

4. ***Double Transpose:***

$$(f^t)^t = f \quad \textit{(via canonical identification with the bidual)}$$

Example 1.5.2 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *be defined by the matrix:*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$$

Then the transpose map $f^t : \mathbb{R}^{2*} \to \mathbb{R}^{3*}$ *is represented by:*

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$$

For any $\varphi = (a, b) \in \mathbb{R}^{2*}$*, we have:*

$$f^t(\varphi) = (a, 2a + b, 3a + 4b)$$

To find $\ker(f^t)$*, solve:*

$$a = 0, \quad b = 0 \Rightarrow \ker(f^t) = \{0\}$$

Thus:

$$\ker(f^t) = (\textit{Im}(f))^\perp = \{0\} \Rightarrow \textit{Im}(f) \textit{ is full-dimensional in } \mathbb{R}^2.$$

1.6 Exercises

In this section, we present a series of application-oriented exercises designed to illustrate the practical relevance of the main theoretical results established earlier.

1.6.1 Exercises Statements

Exercise 1.6.1 *Show that the application which assigns the trace of a matrix in the vector space* $\mathcal{M}_n$ *of square matrices of order* n *over a field* $\mathbb{K}$ *is a linear form.*

Exercise 1.6.2 *Is the determinant function from* $\mathcal{M}_n$ *to* $\mathbb{K}$ *a linear form?*

Exercise 1.6.3 *Let* $E = \mathbb{R}^3$*. In each of the following cases, show that the given vectors form a basis of* E *and compute their dual basis.*

1. $e_1 = (1, 0, -1),\ e_2 = (-1, -1, 2),\ e_3 = (-2, 1, -2)$
2. $e_1 = (1, 0, 0),\ e_2 = (1, 1, 0),\ e_3 = (0, 1, 1)$
3. $e_1 = (1, 1, 0),\ e_2 = (0, 1, 1),\ e_3 = (1, 1, 1)$

Exercise 1.6.4 *Let* $E = \mathbb{R}^3$*. In each of the following cases, show that the given linear forms form a basis of* E^* *and compute their predual basis.*

1. $f_1 = (1, 1, -1),\ f_2 = (1, -1, 1),\ f_3 = (1, 1, 1)$
2. $f_1 = (1, 2, 3),\ f_2 = (2, 3, 4),\ f_3 = (3, 4, 6)$
3. $f_1 = (0, 1, 1),\ f_2 = (1, 0, 1),\ f_3 = (1, 1, 0)$

Exercise 1.6.5

1. *Determine the linear form* f *on* $\mathbb{R}^3$ *such that:*

$$f(1, 1, 1) = 0, \quad f(2, 0, 1) = 1, \quad f(1, 2, 3) = 4$$

2. *Give a basis of the kernel of* f*.*

Exercise 1.6.6 *Let* $f_1, f_2 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ *defined by:*

$$f_1(x, y) = x + y, \quad f_2(x, y) = x - y$$

1. *Show that* $\{f_1, f_2\}$ *forms a basis of* $(\mathbb{R}^2)^*$.

2. *Express the following linear forms in the basis* $\{f_1, f_2\}$*:*

$$g(x, y) = x, \quad h(x, y) = x - 3y$$

Exercise 1.6.7 *Let* E *be a real vector space of dimension 3, with basis* (e_1, e_2, e_3)*. Let* f_1^*, f_2^*, f_3^* *be the linear forms on* E *defined by:*

$$f_1^* = 2e_1^* + e_2^* + e_3^*, \quad f_2^* = -e_1^* + 2e_3^*, \quad f_3^* = e_1^* + 3e_2^*$$

Show that $\{f_1^*, f_2^*, f_3^*\}$ *is a basis of* E^* *and determine the basis* $\{f_1, f_2, f_3\}$ *of* E *for which it is the dual basis.*

Exercise 1.6.8 *Let* E *be the vector space of real polynomials of degree less than or equal to 2. Let* $B = \{1, X, X^2\}$ *be a basis of E, and* B^* *its dual basis. Consider the following linear forms on* E*:*

$$\ell_1(P) = P(1), \quad \ell_2(P) = P'(1), \quad \ell_3(P) = \int_0^1 P(x)\, dx$$

1. *What are the coordinates of* ℓ_1, ℓ_2, ℓ_3 *in* B^**?*

2. *Show that* (ℓ_1, ℓ_2, ℓ_3) *is a basis of the dual space* E^*.

3. *Find a basis of E for which* (ℓ_1, ℓ_2, ℓ_3) *is the dual basis.*

Exercise 1.6.9 *Let* $V = \mathbb{R}_1[X]$ *be the space of real polynomials of degree at most 1. Consider the following applications:*

$$Q_1 : V \to \mathbb{R}, \quad Q_1(f(t)) = \int_0^1 f(t)\, dt$$

$$Q_2 : V \to \mathbb{R}, \quad Q_2(f(t)) = \int_0^2 f(t)\, dt$$

1. *Show that these are linear forms.*

2. *Find the basis* $\{e_1, e_2\}$ *of V for which* $\{Q_1, Q_2\}$ *is the dual basis.*

Exercise 1.6.10 *Let W be the subspace of $\mathbb{R}^4$ generated by the family:*

$$S = \{v_1 = (1, 2, -3, 4),\ v_2 = (0, 1, 4, -1)\}$$

Find a basis of the annihilator of W.

Exercise 1.6.11 *Let v be the linear form on $\mathbb{R}^2$ defined by:*

$$v(x, y) = 3x - 2y$$

For any linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$, what is the transpose ${}^t f$ of f?

1.6.2 Exercises Corrections

Solution 1.6.1 *Let Tr $: M_n(\mathbb{K}) \to \mathbb{K}$ be the function that maps a matrix $A = (a_{ij}) \in M_n(\mathbb{K})$ to its trace:*

$$\textit{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

We want to show that Tr is a linear form, i.e., a linear map from the vector space $M_n(\mathbb{K})$ to the field $\mathbb{K}$.
Let $A, B \in M_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$. Then:

$$\textit{Tr}(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \textit{Tr}(A) + \textit{Tr}(B),$$

$$\textit{Tr}(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda \textit{Tr}(A).$$

Therefore, Tr is additive and homogeneous, hence linear. Thus, the trace function is a linear form on $M_n(\mathbb{K})$.

Solution 1.6.2 *Let* $\det : M_n(\mathbb{K}) \to \mathbb{K}$ *be the determinant function. We want to determine whether this function is linear.*
Recall that the determinant is a multilinear and alternating function of the rows (or columns) of a matrix, but not linear in the entire matrix space. Specifically, for matrices $A, B \in M_n(\mathbb{K})$ and scalar $\lambda \in \mathbb{K}$, in general:

$$\det(A + B) \neq \det(A) + \det(B), \quad \textit{and} \quad \det(\lambda A) = \lambda^n \det(A).$$

The second property shows that homogeneity fails unless $\lambda = 1$ *or* $n = 1$. *Therefore, the determinant is not a linear form on* $M_n(\mathbb{K})$.
Thus The determinant function is not linear on $M_n(\mathbb{K})$.

Solution 1.6.3 *1.*

$$e_1 = (1, 0, -1), \quad e_2 = (-1, -1, 2), \quad e_3 = (-2, 1, -2)$$

Let $A = [e_1 \; e_2 \; e_3]$ *be the matrix whose columns are the vectors. Compute its determinant:*

$$\det(A) = \begin{vmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ -1 & 2 & -2 \end{vmatrix} = 3 \neq 0$$

So the vectors form a basis. To find the dual basis, compute A^{-1}*, then transpose to get the dual basis:*

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 0 & -6 & -3 \\ -1 & -4 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \quad (e_1^*, e_2^*, e_3^*) = (A^{-1})^t$$

Thus, the dual basis vectors are:

$$e_1^* = (0, -2, -1), \quad e_2^* = \left(-\frac{1}{3} - \frac{4}{3}, -\frac{1}{3}\right), \quad e_3^* = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

2.

$$e_1 = (1, 0, 0), \quad e_2 = (1, 1, 0), \quad e_3 = (0, 1, 1)$$

Matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(A) = 1 \neq 0$$

Inverse:

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (e_1^*, e_2^*, e_3^*) = (A^{-1})^t$$

Dual basis:

$$e_1^* = (1, -1, 1), \quad e_2^* = (0, 1, -1), \quad e_3^* = (0, 0, 1)$$

3.

$$e_1 = (1,1,0), \quad e_2 = (0,1,1), \quad e_3 = (1,1,1)$$

Matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \det(A) = 1 \neq 0$$

Inverse:

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad (e_1^*, e_2^*, e_3^*) = (A^{-1})^t$$

Dual basis:

$$e_1^* = (0,1,-1), \quad e_2^* = (-1,1,0), \quad e_3^* = (1,-1,1)$$

Solution 1.6.4 *1.*

$$f_1 = (1,1,-1), \quad f_2 = (1,-1,1), \quad f_3 = (1,1,1)$$

Matrix:

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \det(F) = -4 \neq 0$$

Inverse:

$$F^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad (f_1^*, f_2^*, f_3^*) = (F^{-1})^t$$

Predual basis:

$$f_1^* = \left(\frac{1}{2}, 0, -\frac{1}{2}\right), \quad f_2^* = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \quad f_3^* = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

2.

$$f_1 = (1,2,3), \quad f_2 = (2,3,4), \quad f_3 = (3,4,6)$$

Matrix:

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}, \quad \det(F) = -1 \neq 0$$

Inverse:

$$F^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \quad (f_1^*, f_2^*, f_3^*) = (F^{-1})^t$$

Predual basis:

$$f_1^* = (-2, 0, 1), \quad f_2^* = (0, 3, -2), \quad f_3^* = (1, -2, 1)$$

3.

$$f_1 = (0, 1, 1), \quad f_2 = (1, 0, 1), \quad f_3 = (1, 1, 0)$$

Matrix:

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \det(F) = 2 \neq 0$$

Inverse:

$$F^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad (f_1^*, f_2^*, f_3^*) = (F^{-1})^t$$

Predual basis:

$$f_1^* = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \quad f_2^* = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \quad f_3^* = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$$

Solution 1.6.5 *We know that every linear form on the vector space $\mathbb{K}^n$ is of the form*

$$a_1 x_1 + \cdots + a_n x_n.$$

Thus, the linear form f is given by:

$$f(x, y, z) = ax + by + cz.$$

We are given the following conditions:

$$\begin{cases} f(1,1,1) = 0 \\ f(2,0,1) = 1 \\ f(1,2,3) = 4 \end{cases}$$

This leads to the following system of equations:

$$\begin{cases} x + y + z = 0 \quad \textit{(1)} \\ 2x + z = 1 \quad \textit{(2)} \\ x + 2y + 3z = 4 \quad \textit{(3)} \end{cases}$$

From equation (2), we isolate z*:*

$$z = -2x + 1.$$

Substituting into equation (1), we get:

$$x + y + (-2x + 1) = 0 \Rightarrow y = x - 1.$$

Substituting into equation (3), we get:

$$x + 2(x - 1) + 3(-2x + 1) = 4 \Rightarrow x = -1.$$

Thus, we find:

$$x = -1, \quad y = 2, \quad z = 3.$$

Therefore, the linear form is:

$$f(x, y, z) = -x - 2y + 3z.$$

To find a basis of the kernel of f*, we consider:*

$$\ker f = \Big\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\Big\}.$$

That is,

$$f(x, y, z) = 0 \Leftrightarrow -x - 2y + 3z = 0 \Leftrightarrow x = 3z - 2y.$$

So,

$$\ker f = \{(3z - 2y, y, z) \mid y, z \in \mathbb{R}\} = \{y(-2, 1, 0) + z(3, 0, 1) \mid y, z \in \mathbb{R}\}.$$

Hence, the set $\{v_1 = (-2, 1, 0),\ v_2 = (3, 0, 1)\}$ *is a generating set of* $\ker f$*. Since* v_1 *and* v_2 *are linearly independent, they form a basis of* $\ker f$*.*

Solution 1.6.6 *1. The set* $\{f_1, f_2\}$ *forms a basis of* $\mathbb{R}^{2*}$*. Indeed, since the family contains two elements (the same as* $\dim \mathbb{R}^{2*}$*), it suffices to show that* f_1 *and* f_2 *are linearly independent. Let* $\alpha, \beta \in \mathbb{R}$ *such that:*

$$\alpha f_1 + \beta f_2 = 0 \Rightarrow \alpha(x+y) + \beta(x-y) = 0.$$

Then:

$$\alpha(x+y) + \beta(x-y) = x(\alpha+\beta) + y(\alpha-\beta) = 0 \quad \textit{for all } (x,y) \in \mathbb{R}^2.$$

This implies the system:

$$\begin{cases} \alpha+\beta = 0 \\ \alpha-\beta = 0 \end{cases} \Rightarrow \alpha = \beta = 0.$$

Therefore, the family $\{f_1, f_2\}$ *is linearly independent and hence a basis of* $\mathbb{R}^{2*}$*.*

Another way to answer this question is to write the matrix whose rows are the row vectors associated with the linear forms f_1 *and* f_2*. The family is linearly independent if the associated matrix has rank equal to* $2 = \dim \mathbb{R}^{2*}$*.*

2.
- *For* $g(x,y) = x$*, we have:*

  $$f_1 + f_2 = (x+y) + (x-y) = 2x = 2g \Rightarrow g = \frac{1}{2}f_1 + \frac{1}{2}f_2.$$

- *For* $h(x,y) = x - 3y$*, we seek* $\alpha, \beta \in \mathbb{R}$ *such that* $h = \alpha f_1 + \beta f_2$*, i.e., for all* $(x,y) \in \mathbb{R}^2$*:*

  $$h(x,y) = \alpha f_1(x,y) + \beta f_2(x,y) = \alpha(x+y) + \beta(x-y).$$

  Then:

  $$x - 3y = \alpha(x+y) + \beta(x-y) = x(\alpha+\beta) + y(\alpha-\beta).$$

  This gives the system:

  $$\begin{cases} \alpha+\beta = 1 \\ \alpha-\beta = -3 \end{cases} \Rightarrow \alpha = -1, \quad \beta = 2.$$

  Therefore, $h = -f_1 + 2f_2$*.*

Solution 1.6.7 *To show that the family $\{f_1^*, f_2^*, f_3^*\}$ is a basis of E^*, it suffices to prove that it is linearly independent. Alternatively, we can write the matrix representing these linear forms in the dual basis $\{e_1^*, e_2^*, e_3^*\}$ and prove that its rank is equal to 3. Let M be the matrix whose rows are the coordinates of the linear forms f_1^*, f_2^*, and f_3^* in the basis $\{e_1^*, e_2^*, e_3^*\}$:*

$$M = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

We compute the determinant:

$$\det M = -13 \neq 0 \Rightarrow \operatorname{rank}(M) = 3.$$

Therefore, the family $\{f_1^*, f_2^*, f_3^*\}$ is linearly independent and hence forms a basis of E^*. Let $\{f_1, f_2, f_3\}$ be the basis of E such that $\{f_i^*\}$ is its dual basis. Suppose:

$$f_1 = ae_1 + be_2 + ce_3.$$

We obtain the following system:

$$\begin{cases} f_1^*(f_1) = 1 \\ f_2^*(f_1) = 0 \\ f_3^*(f_1) = 0 \end{cases} \Rightarrow \begin{cases} 2a + b + c = 1 \quad \textit{(1)} \\ -a + 2c = 0 \quad \textit{(2)} \\ a + 3b = 0 \quad \textit{(3)} \end{cases}$$

From (2): $c = \frac{1}{2}a$, and from (3): $b = -\frac{1}{3}a$. Substituting into (1):

$$2a - \frac{1}{3}a + \frac{1}{2}a = 1 \Rightarrow \frac{13}{6}a = 1 \Rightarrow a = \frac{6}{13}.$$

Then:

$$b = -\frac{2}{13}, \quad c = \frac{3}{13}.$$

So:

$$f_1 = \frac{6}{13}e_1 - \frac{2}{13}e_2 + \frac{3}{13}e_3.$$

Following the same procedure for f_2 and f_3, we obtain:

$$\begin{cases} f_2 = -\frac{3}{13}e_1 + \frac{1}{13}e_2 + \frac{5}{13}e_3 \\ f_3 = -\frac{2}{13}e_1 + \frac{5}{13}e_2 - \frac{1}{13}e_3 \end{cases}$$

We can observe that the coordinates of the vectors f_i expressed in the basis $\{e_i\}$ are precisely the column vectors of the inverse matrix M^{-1}.

Indeed, we have:

$$M^{-1} = \frac{1}{13}\begin{pmatrix} 6 & -3 & -2 \\ -2 & 1 & 5 \\ 3 & 5 & -1 \end{pmatrix}.$$

Solution 1.6.8 *1. Let $E = \mathbb{R}_2[X]$, the space of real polynomials of degree less than or equal to 2. Let $B = \{1, X, X^2\}$ be a basis of E, and $B^* = \{b_1^*, b_2^*, b_3^*\}$ its dual basis, defined by:*

$$b_i^*(b_j) = \delta_{ij}.$$

We are given three linear forms on E:

$$\ell_1(P) = P(1), \quad \ell_2(P) = P'(1), \quad \ell_3(P) = \int_0^1 P(x)\,dx.$$

Let $P(x) = a_0 + a_1x + a_2x^2 \in E$. Then:

$$\begin{aligned} \ell_1(P) &= P(1) = a_0 + a_1 + a_2, \\ \ell_2(P) &= P'(1) = a_1 + 2a_2, \\ \ell_3(P) &= \int_0^1 P(x)\,dx = \int_0^1 (a_0 + a_1x + a_2x^2)\,dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3}. \end{aligned}$$

Coordinates of ℓ_1, ℓ_2, ℓ_3 in the dual basis B^*

We express each ℓ_i as a linear combination of b_1^*, b_2^*, b_3^*, i.e., find the coordinates of ℓ_i in the dual basis.

Let $P(x) = a_0 + a_1x + a_2x^2$. Then $\ell_i(P) = \ell_i(a_0 \cdot 1 + a_1 \cdot X + a_2 \cdot X^2)$, so each ℓ_i corresponds to a row vector acting on the coordinates (a_0, a_1, a_2).

Thus, the coordinate vectors are:

$$[\ell_1]_{B^*} = (1, 1, 1), \quad [\ell_2]_{B^*} = (0, 1, 2), \quad [\ell_3]_{B^*} = \left(1, \frac{1}{2}, \frac{1}{3}\right)$$

2. *Show that (ℓ_1, ℓ_2, ℓ_3) is a basis of E^*.*

   We check linear independence by computing the determinant of the matrix formed by these *coordinate vectors:*

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \quad \det(M) = \frac{1}{3} \neq 0$$

Therefore, the forms ℓ_1, ℓ_2, ℓ_3 are linearly independent and span E^*, which is 3-dimensional. So they form a basis of E^*.

3. *Find a basis of E for which (ℓ_1, ℓ_2, ℓ_3) is the dual basis*

   Let $B' = \{f_1, f_2, f_3\} \subset E$ be the basis such that $\ell_i(f_j) = \delta_{ij}$. Then B' is the predual basis of (ℓ_1, ℓ_2, ℓ_3).

   We solve for $f_j = a_0 + a_1 x + a_2 x^2$ such that:

   - $\ell_1(f_1) = 1, \ell_2(f_1) = 0, \ell_3(f_1) = 0$
   - $\ell_1(f_2) = 0, \ell_2(f_2) = 1, \ell_3(f_2) = 0$
   - $\ell_1(f_3) = 0, \ell_2(f_3) = 0, \ell_3(f_3) = 1$

   Let $f_j(x) = a_0 + a_1 x + a_2 x^2$. Then the system becomes:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \textit{(for } f_1\textit{)}$$

   Solving this system gives:

$$f_1(x) = -2 + 6x - 3x^2, \quad f_2(x) = \frac{1}{2} - 2x + \frac{3}{2}x^2, \quad f_3(x) = 3 - 6x + 3x^2$$

   Thus, the basis $B' = \{f_1, f_2, f_3\}$ satisfies $\ell_i(f_j) = \delta_{ij}$, and is the basis of E for which (ℓ_1, ℓ_2, ℓ_3) is the dual basis.

Solution 1.6.9 1. *Let $f, g \in V$, and $\lambda \in \mathbb{R}$. Then:*

$$Q_1(f+g) = \int_0^1 (f(t) + g(t))\, dt = \int_0^1 f(t)\, dt + \int_0^1 g(t)\, dt = Q_1(f) + Q_1(g),$$

and

$$Q_1(\lambda f) = \lambda \int_0^1 f(t)\, dt = \lambda Q_1(f).$$

Same applies for Q_2. Hence, both Q_1 and Q_2 are linear forms.

2. *Let* $B = \{1, X\}$ *be the canonical basis of* V. *We express* Q_1 *and* Q_2 *in terms of the dual basis* $B^* = \{b_1^*, b_2^*\}$*, where:*

$$b_1^*(f) = \textit{coefficient of } 1, \quad b_2^*(f) = \textit{coefficient of } X$$

Let $f(t) = a + bt \in V$. *Then:*

$$Q_1(f) = \int_0^1 (a + bt)\, dt = a + \frac{b}{2}, \quad Q_2(f) = \int_0^2 (a + bt)\, dt = 2a + 2b$$

So the coordinate vectors of Q_1 *and* Q_2 *in* B^* *are:*

$$[Q_1]_{B^*} = \left(1, \frac{1}{2}\right), \quad [Q_2]_{B^*} = (2, 2)$$

Let $M = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 2 \end{bmatrix}$. *We want to find a basis* $\{e_1, e_2\} \subset V$ *such that:*

$$Q_i(e_j) = \delta_{ij}$$

This means $\{e_1, e_2\}$ *is the predual basis of* $\{Q_1, Q_2\}$. *Solve:*

$$M \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{(for } e_1 = a + bX\textit{)}$$

$$M \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textit{(for } e_2 = a + bX\textit{)}$$

Compute M^{-1}*:*

$$\det(M) = 1 \cdot 2 - 2 \cdot \frac{1}{2} = 2 - 1 = 1$$

$$M^{-1} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

Then:

$$e_1 = 2 - 2X, \quad e_2 = -\frac{1}{2} + X$$

Thus, the basis $\{e_1, e_2\} = \{2 - 2X, -\frac{1}{2} + X\}$ *is the basis of V for which* $\{Q_1, Q_2\}$ *is the dual basis.*

Solution 1.6.10 *Let* $\ell = (a, b, c, d) \in (\mathbb{R}^4)^*$*. Then:*

$$\ell(v_1) = a + 2b - 3c + 4d = 0 \quad \textit{(1)}$$

$$\ell(v_2) = b + 4c - d = 0 \quad \textit{(2)}$$

From equation (2), solve for b*:*

$$b = d - 4c$$

Substitute into equation (1):

$$a + 2(d - 4c) - 3c + 4d = 0$$

$$a + 2d - 8c - 3c + 4d = 0$$

$$a + 6d - 11c = 0 \Rightarrow a = 11c - 6d$$

Therefore, the general solution is:

$$(a, b, c, d) = (11c - 6d, d - 4c, c, d)$$

Let us express this as a linear combination of two free parameters c *and* d*:*

- *For* $c = 1, d = 0$*:* $(11, -4, 1, 0)$
- *For* $c = 0, d = 1$*:* $(-6, 1, 0, 1)$

Thus, a basis of the annihilator $W^\perp \subset (\mathbb{R}^4)^*$ *is:*

$$\{(11, -4, 1, 0),\ (-6, 1, 0, 1)\}$$

Solution 1.6.11 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *be a linear map represented by the matrix:*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then for any $u = (x, y, z) \in \mathbb{R}^3$*, we have:*

$$f(u) = Au = (a_{11}x + a_{12}y + a_{13}z,\ a_{21}x + a_{22}y + a_{23}z)$$

The transpose of f*, denoted* ${}^t f : (\mathbb{R}^2)^* \to (\mathbb{R}^3)^*$*, satisfies:*

$$({}^t f(v))(u) = v(f(u)) = (3a_{11} - 2a_{21})x + (3a_{12} - 2a_{22})y + (3a_{13} - 2a_{23})z$$

Therefore, the linear form ${}^t f(v) \in (\mathbb{R}^3)^*$ *is given by:*

$${}^t f(v) = (3a_{11} - 2a_{21},\ 3a_{12} - 2a_{22},\ 3a_{13} - 2a_{23})$$

Conclusion

Linear forms and the dual space deepen our understanding of vector spaces by showing how vectors can be studied via the functions that measure them. Duality supplies practical tools for representing linear transformations, choosing convenient bases, and solving problems more cleanly. Mastering these notions prepares the way for advanced topics where the dual perspective is essential.

Chapter 2

Bilinear forms on a vector space

Introduction

Bilinear forms are fundamental objects in linear algebra that extend the concept of scalar products to more general settings. Defined as functions that take two vectors and return a scalar, bilinear forms capture the interaction between vectors in a way that is linear in each argument. In this chapter we present a basic introduction on Bilinear forms on a vector space including rank, kernel, Orthogonalization of Gram-Schmidt, Orthogonal matrices of real symmetric matrices see [2, 4].

2.1 Bilinear forms

2.1.1 Definitions

In this section, $\mathbb{R}$ is the field of real numbers and E is a vector space over $\mathbb{R}$. For example, $E = \mathbb{R}^n$, $\mathbb{R}^n[x]$ or $P_n[x]$, $C([a,b],\mathbb{R})$, $C^\infty([a,b],\mathbb{R})$, and $\mathcal{M}_n(\mathbb{R})$ with $n \geq 1$, and so on.

Let E be a vector space on $\mathbb{R}$. As above, a linear form is a mapping f from E to $\mathbb{R}$ such that for every $(x,y) \in E^2$ and $\lambda \in \mathbb{R}$, we have

(i) $f(x+y) = f(x) + f(y)$;

(ii) $f(\lambda x) = \lambda f(x)$.

Similarly, we have the following definition:

Definition 2.1.1 *Let E be a vector space on $\mathbb{R}$. A bilinear form is a mapping f from E^2 to $\mathbb{R}$ such that for every $(x, \grave{x}, y, \grave{y}) \in E^4$ and $\lambda \in \mathbb{R}$, one has*

(i) $f(\lambda x + \grave{x}, y) = \lambda f(x, y) + f(\grave{x}, y)$ *;*

(ii) $f(x, \lambda y + \grave{y}) = \lambda f(x, y) + f(x, \grave{y})$.

As in (1.2) and (1.3), note that a bilinear form is a mapping f from E^2 to $\mathbb{R}$ such that f is linear from the left and linear from the right. For details, we present the following remark.

Remark 2.1.1 *Let $f : E \times E \to \mathbb{R}$ be a bilinear form on E. This means that for all $(x, \grave{x}, y, \grave{y}) \in E^4$ and $\lambda \in \mathbb{R}$ we have*

- $f(x + \grave{x}, y) = f(x, y) + f(\grave{x}, y)$,
- $f(x, y + \grave{y}) = f(x, y) + f(x, \grave{y})$,
- $f(\lambda x, y) = \lambda f(x, y)$,
- $f(x, \lambda y) = \lambda f(x, y)$.

Proposition 2.1.1 *The set of all bilinear forms from $E \times F$ to $\mathbb{K}$, denoted $\mathcal{L}(E \times F, \mathbb{K})$ is a vector space over $\mathbb{K}$.*

Proof. Let $\alpha, \beta \in \mathbb{K}$ and $f, g, h \in \mathcal{L}(E \times F, \mathbb{K})$, where $\mathcal{L}(E \times F, \mathbb{K})$ denotes the set of bilinear forms from $E \times F$ to $\mathbb{K}$.

1. **(a)** The zero bilinear form

   Define:

   $$f_0 : E \times F \to K, \quad (x, y) \mapsto 0_{\mathbb{K}}$$

   This is the neutral element of $\mathcal{L}(E \times F, \mathbb{K})$ under addition.

   (b) Associativity:

   For all $f, g, h \in \mathcal{L}(E \times F, \mathbb{K})$, we have:

   $$(f + g) + h = f + (g + h)$$

   So addition is associative.

(c) Existence of additive inverse:

For any $f \in \mathcal{L}(E \times F, \mathbb{K})$, define:

$$(-f)(x,y) = -f(x,y)$$

then:

$$f + (-f) = f_{0_{\mathcal{L}(E\times F,\mathbb{K})}}$$

So every element has an additive inverse.

(d) Commutativity:

For all $f, g \in \mathcal{L}(E \times F, \mathbb{K})$, we have:

$$f + g = g + f$$

So addition is commutative.

Then, $(\mathcal{L}(E \times F, \mathbb{K}), +)$ is an abelian group.

2. $(\alpha\beta) \cdot g = \alpha \cdot (\beta \cdot g)$
3. $(\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f$
4. $\alpha \cdot (f + g) = \alpha \cdot f + \alpha \cdot g$
5. $1_{\mathbb{K}} \cdot f = f$.

Then, $\mathcal{L}(E \times F, \mathbb{K})$ is a vector space over $\mathbb{K}$. ■

2.1.2 Matrix of a bilinear form

Let E and F be two finite-dimensional vectors $\mathbb{K}$-space with $B = (e_1, ..., e_n)$ a base of E and $\grave{B} = (\epsilon_1, ..., \epsilon_p)$ a base of F

Definition 2.1.2 *Let $f \in \mathcal{L}(E \times F, \mathbb{K})$. The matrix associated with f with respect to B and $\grave{B}$ is the matrix of type (n, p) defined by: $a_{ij} = f(e_i, \epsilon_j)$, $\forall i = \overline{1,n}$, $\forall j = \overline{1,p}$, i.e*

$$M = \begin{pmatrix} f(e_1,\epsilon_1) & f(e_1,\epsilon_2) & \dots & f(e_1,\epsilon_p) \\ f(e_2,\epsilon_1) & f(e_2,\epsilon_2) & \dots & f(e_2,\epsilon_p) \\ \vdots & \vdots & \ddots & \vdots \\ f(e_n,\epsilon_1) & f(e_n,\epsilon_2) & \dots & f(e_n,\epsilon_p) \end{pmatrix}$$

Example 2.1.1 *Let the bilinear form*

$$\begin{array}{rcl} f:\mathbb{R}^3\times\mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ ((x_1,x_2,x_3),(y_1,y_2)) & \longmapsto & (x_1+x_2+x_3)y_1+(x_1+x_2+x_3)y_2 \end{array}$$

and let $B=\{e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)\}$ *the canonical base of* $\mathbb{R}^3$ *and* $\grave{B}=\{\epsilon_1=(1,0),\epsilon_2=(1,1)\}$ *a base of* $\mathbb{R}^2$.

The matrix associated with f *with respect to* B *and* $\grave{B}$ *is:*

$$M=\begin{pmatrix} f(e_1,\epsilon_1) & f(e_1,\epsilon_2) \\ f(e_2,\epsilon_1) & f(e_2,\epsilon_2) \\ f(e_3,\epsilon_1) & f(e_3,\epsilon_2) \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

2.1.3 Bilinear form associated with a matrix

Definition 2.1.3 *Let* $A\in\mathcal{M}_{n,p}(\mathbb{R})$ *be a matrix. The application*

$$\begin{array}{rcl} f:\mathbb{R}^n\times\mathbb{R}^p & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & X^tAY. \end{array}$$

Or $X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix},$

is a bilinear form called the bilinear form associated with A.

Example 2.1.2 *Let the matrix* $A=\begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 4 \end{pmatrix}$.

The bilinear form associated with A *is:*

$$\begin{array}{rcl} f:\mathbb{R}^2\times\mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ ((x_1,x_2),(y_1,y_2,y_3)) & \longmapsto & f((x_1,x_2),(y_1,y_2,y_3)); \end{array}$$

with

$$\begin{aligned} f((x_1,x_2),(y_1,y_2,y_3)) &= (x_1,x_2)\begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= 2x_1y_1 + (3x_1 - 2x_2)y_2 + (x_1 + 4x_2)y_3 \end{aligned}$$

2.1.4 The base-change formula

Theorem 2.1.1 *Let* $f \in \mathcal{L}(E \times F, \mathbb{K})$*,* $B = \{e_1, e_2, \ldots, e_n\}$ *a basis of* E*, and* $B' = \{e'_1, e'_2, \ldots, e'_p\}$ *a basis of* F*, and let* M *be the matrix associated with* f *with respect to* B *and* B'*. Now let* $B_1 = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ *be another basis of* E*, and* $B'_1 = \{\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_p\}$ *another basis of* F*, and let* N *be the matrix associated with* f *with respect to* B_1 *and* B'_1*. Then:*

$$N = {}^tPMQ$$

where:

P *is the change-of-basis matrix from* B *to* B_1*,*

Q *is the change-of-basis matrix from* B' *to* B'_1*.*

Proof. Using matrix notation with respect to the bases B, B' and B_1, B'_1, we have:

$$\forall (X,Y) \in E \times F : f(X,Y) = {}^tXMY = {}^tX'NY'$$

with:

$$X = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x'_i \epsilon_i, \quad Y = \sum_{i=1}^{p} y_i e'_i = \sum_{i=1}^{p} y'_i \epsilon'_i.$$

Let P be the change-of-basis matrix from B to B_1, so $X = PX'$.

Let Q be the change-of-basis matrix from B' to B'_1, so $Y = QY'$.

Therefore:

$${}^tXMY = {}^t(PX')M(QY') = {}^tX'{}^tPMQY = {}^tX'({}^tPMQ)Y$$

Thus, for all $X' \in \mathbb{K}^n, Y' \in \mathbb{K}^p$, we have:

$${}^tX'NY' = {}^tX'({}^tPMQ)Y$$

So:

$$N = {}^tPMQ$$

■

Example 2.1.3 *Let* f *be a bilinear form define on* $\mathbb{R}^3$ *by:*

$$f(x, y) = x_1y_2 + x_2y_3 + x_3y_3.$$

Let $B = \{e_1 = (1, 1, -1), e_1 = (1, -1, 0), e_1 = (0, 1, 1)\}$ *base on* $\mathbb{R}^3$.

We have

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, M = [f]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then the matrix associated of f *is:*

$$\begin{aligned} P^tMP &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \end{aligned}$$

2.2 Symetric bilinear form

Definition 2.2.1 *Let* $f : E^2 \to \mathbb{K}$ *be a bilinear form.*

1. f *is said to be* ***symmetric*** *if*

$$\forall (x, y) \in E^2, f(x, y) = f(y, x).$$

2. f *is said to be* ***antisymmetric*** *if*

$$\forall (x, y) \in E^2, f(x, y) = -f(y, x).$$

3. f *is said to be* ***alternating*** *if*

$$\forall x \in E, f(x, x) = 0.$$

Examples 2.2.1 *We can easily check that the following mappings are symmetric bilinear forms.*

1. $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, f(x, y) = xy.$
2. $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, ((x, y), (\grave{x}, \grave{y})) \longmapsto x\grave{x} + y\grave{y}.$
3. $\phi : P[x] \times P[x] \to \mathbb{R}$ *with* $\phi(p, q) = \int_a^b p(t)q(t)dt.$
4. *Let* $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ *with* $f(x, y) = x_1y_1 + x_2y_2 - x_3y_3.$

Example 2.2.1 *Let* $u = (x, y)$ *,* $v = (\grave{x}, \grave{y}) \in \mathbb{R}^2$ *with* $f(u, v) = x\grave{y} - \grave{x}y$. *Then* f *is a antisymmetric alternating bilinear form.*

Proposition 2.2.1 **i)** *The set of symmetric bilinear forms, denoted* $\mathcal{S}(E \times F, \mathbb{K})$*, is a vector subspace of* $\mathcal{L}(E \times F, \mathbb{K})$*.*

ii) *The set of antisymmetric bilinear forms, denoted* $\mathcal{A}(E \times F, \mathbb{K})$*, is a vector subspace of* $\mathcal{L}(E \times F, \mathbb{K})$*.*

Proof.

i) a) We have $f_0(x, y) = f_0(y, x) = 0_{\mathbb{K}}$, so

$$f_0 \in \mathcal{S}(E \times E, \mathbb{K}). \tag{2.1}$$

b) Let $f, g \in \mathcal{S}(E \times E, \mathbb{K})$ and $a, b \in \mathbb{K}$:

$$\begin{aligned}(af + bg)(x, y) &= af(x, y) + bg(x, y)\\ &= af(y, x) + bg(y, x)\\ &= (af + bg)(y, x)\end{aligned}$$

Thus,

$$af + bg \in \mathcal{S}(E \times E, \mathbb{K}). \tag{2.2}$$

From (2.1) and (2.2), $\mathcal{S}(E \times E, \mathbb{K})$ is a vector subspace of $\mathcal{L}(E \times E, \mathbb{K})$.

ii) a) We have $f_0(x, y) = -f_0(y, x) = 0_{\mathbb{K}}$, so

$$f_0 \in \mathcal{A}(E \times E, \mathbb{K}). \tag{2.3}$$

b) Let $f, g \in \mathcal{A}(E \times E, \mathbb{K})$ and $a, b \in \mathbb{K}$:

$$\begin{aligned}(af + bg)(x, y) &= af(x, y) + bg(x, y) \\ &= -af(y, x) - bg(y, x) \\ &= -(af + bg)(y, x)\end{aligned}$$

Thus,

$$af + bg \in \mathcal{A}(E \times E, \mathbb{K}). \tag{2.4}$$

From (2.3) and (2.4), $\mathcal{A}(E \times E, \mathbb{K})$ is a vector subspace of $\mathcal{L}(E \times E, \mathbb{K})$.

■

Proposition 2.2.2 *We have*

$$\mathcal{L}(E \times F, \mathbb{K}) = \mathcal{S}(E \times F, \mathbb{K}) \oplus \mathcal{A}(E \times F, \mathbb{K})$$

Proof.

We have $f_0 = 0 \in \mathcal{S}(E \times F, \mathbb{K}) \cap \mathcal{A}(E \times F, \mathbb{K})$. Also, if $f \in \mathcal{S}(E \times F, \mathbb{K}) \cap \mathcal{A}(E \times F, \mathbb{K})$, then by 2.2.1

$$f(x, y) = f(y, x) = -f(y, x) \quad \text{for every } x, y \in E.$$

Hence, $f(x, y) = 0$ for every $x, y \in E$. So, $f = f_0$. Thus, we have proved that:

$$\mathcal{S}(E \times F, \mathbb{K}) \cap \mathcal{A}(E \times F, \mathbb{K}) = \{0_E\}. \tag{2.1}$$

Now, let $f \in \mathcal{L}(E \times F, \mathbb{K})$. For any $x, y \in E$, we see that:

$$f(x, y) = \frac{f(x, y) - f(y, x)}{2} + \frac{f(x, y) + f(y, x)}{2} = h_1(x, y) + h_2(x, y),$$

where $h_1 \in \mathcal{A}(E \times F, \mathbb{K})$ and $h_2 \in \mathcal{S}(E \times F, \mathbb{K})$.
So

$$\mathcal{S}(E \times F, \mathbb{K}) \subset \mathcal{S}(E \times F, \mathbb{K}) + \mathcal{A}(E \times F, \mathbb{K})$$

Thus

$$\mathcal{L}(E \times F, \mathbb{K}) = \mathcal{S}(E \times F, \mathbb{K}) + \mathcal{A}(E \times F, \mathbb{K}). \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\mathcal{L}(E \times F, \mathbb{K}) = \mathcal{S}(E \times F, \mathbb{K}) \oplus \mathcal{A}(E \times F, \mathbb{K}).$$

■

2.3 Rank and kernel of a bilinear form

2.3.1 kernel of a bilinear form

Definition 2.3.1 *The **kernel** of the bilinear form f defined on the vector space E is called the set defined by:*

$$\ker(f) = \{y \in E / f(x, y) = 0, \forall x \in E\}$$

Definition 2.3.2 *A bilinear form f is said to be nondegenerate or regular if its kernel is reduced to the zero vector, i.e.*

$$\ker(f) = \{O_E\}$$

Examples 2.3.1 *1. Let the symmetric bilinear form*

$$\begin{aligned} f : \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R} \\ ((x_1, x_2), (y_1, y_2)) &\mapsto x_1y_1 + x_2y. \end{aligned}$$

We have

$$\ker(f) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \forall (y_1, y_2) \in \mathbb{R}^2,\ f((x_1, x_2), (y_1, y_2)) = 0\}.$$

Thus

$$f((x_1, x_2), (y_1, y_2)) = 0 \quad \textit{for all } (y_1, y_2) \in \mathbb{R}^2.$$

- *For* $(y_1, y_2) = (1, 0)$ *we obtain* $x_1 = 0$.
- *For* $(y_1, y_2) = (0, 1)$ *we obtain* $x_2 = 0$.

Therefore

$$\ker(f) = \{(0,0)\}.$$

Consequently f is nondegenerate.

2. *Let the symmetric bilinear form*

$$\begin{array}{rcl} g : \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ ((x_1,x_2),(y_1,y_2)) & \mapsto & (x_1+x_2)(y_1+y_2). \end{array}$$

We have

$$\ker(g) = \{(x_1,x_2) \in \mathbb{R}^2 \mid \forall (y_1,y_2) \in \mathbb{R}^2,\ g((x_1,x_2),(y_1,y_2)) = 0\}.$$

Thus

$$g((x_1,x_2),(y_1,y_2)) = 0 \ \ for\ all\ (y_1,y_2) \in \mathbb{R}^2.$$

- *For any $(y_1,y_2) \neq (0,0)$ this implies $x_1 = -x_2$.*

Therefore

$$\ker(g) = \{(x_1,-x_1) \mid x_1 \in \mathbb{R}\} = \langle (1,-1) \rangle.$$

Since $\ker(g) \neq \{(0,0)\}$, g is degenerate.

Proposition 2.3.1 *Let $f \in \mathcal{L}(E \times F, \mathbb{K})$. Consider the map $\Phi : E \to E^*$ defined by:*

$$\forall y \in E, \qquad \Phi(y) = \phi_y,\ \forall x \in E, \qquad \phi_y(x) = f(x,y).$$

Then Φ is injective if and only if f is nondegenerate.

Proof. We have that Φ is linear and

$$\begin{array}{rcl} \ker(\Phi) & = & \{x \in E \mid \Phi(x) = 0\} \\ & = & \{x \in E \mid \phi_y(x) = 0\ \forall y \in E\} \\ & = & \{x \in E \mid f(x,y) = 0\ \forall y \in E\} = \ker(f). \end{array}$$

Since Φ is injective, $\ker\Phi = \{0_E\} = \ker(f)$, consequently f is nondegenerate. ■

Corollaire 2.3.1 *Let* $f \in \mathcal{L}(E \times E, \mathbb{K})$*, where* E *is finite-dimensional. Let* B *be a basis of* E *and let* A *be the matrix of* f *with respect to the basis* B*. Then*

$$f \textit{ is nondegenerate } \Leftrightarrow \det(A) \neq 0.$$

Proof. Consider the map $\Phi : E \to E^*$ defined by:

$$\forall y \in E, \quad \Phi(y) = \phi_y, \; \forall x \in E, \quad \phi_y(x) = f(x, y).$$

Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of E and let $B^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$ be the dual basis of B. Let $M = (m_{ij})_{1 \leq i,j \leq n}$ be the matrix of Φ with respect to the bases B and B^*. Then for each $j = 1, \ldots, n$, we have

$$\Phi(e_j) = \sum_{k=1}^{n} m_{kj}\, e_k^*.$$

Therefore for each $i, j \in \{1, \ldots, n\}$, we have

$$f(e_i, e_j) = \Phi(e_i)(e_j) = \sum_{k=1}^{n} m_{kj}\, e_k^*(e_j) = m_{ij}, \quad \textit{since} \;\; e_k^*(e_i) = \delta_{ki}.$$

We therefore deduce that $M = A$. Hence

$$\begin{aligned} f \text{ is nondegenerate } &\Leftrightarrow \Phi \text{ is injective (proposotion(2.3.1))} \\ &\Leftrightarrow \Phi \text{ is bijective (since } \dim E = \dim E^*) \\ &\Leftrightarrow \det(M) \neq 0 \\ &\Leftrightarrow \det(A) \neq 0, \text{ since } M = A. \end{aligned}$$

■

2.3.2 Rank of a bilinear form

Definition 2.3.3 *The* ***rank*** *of a bilinear form* f *defined on a finite-dimensional vector space* E *of dimension* n*, denoted* ***rg(f)****, is the rank of its associated matrix.*

Remark 2.3.1 *A bilinear form is nondegenerate if its rank is maximal, i.e.*

$$rg(f) = n = \dim(E).$$

Example 2.3.1 *Let the bilinear form $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, defined in the canonical basis by:*

$$f(x,y) = x_1y_1 + x_1y_2 - x_1y_3 + x_2y_1 - 3x_2y_3 - x_3y_1 - 3x_3y_2 - 3x_3y_3.$$

Its matrix is:

$$M = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & -3 \end{pmatrix}$$

Compute the determinant:

$$\det(M) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & -3 \end{vmatrix} = 0$$

Since $\det(M) = 0$*, we have rank*$(M) < 3$*, and the form f is degenerate.*

2.4 Orthogonal matrices

Definition 2.4.1 *Let $f \in \mathcal{L}(E \times E, \mathbb{K})$. Two vectors x and y in E are said to be orthogonal if*

$$f(x,y) = 0.$$

Definition 2.4.2 *Let $f \in \mathcal{L}(E \times E, \mathbb{K})$ and let A be a nonempty subset of E. The orthogonal of A with respect to f, denoted $A^\perp$, is*

$$A^\perp = \{x \in E : f(x,y) = 0 \textit{ for all } y \in A\}.$$

Example 2.4.1 *Let $f \in \mathcal{L}(E \times E, \mathbb{K})$.*

1. $E^\perp = \{x \in E : f(x,y) = 0 \textit{ for all } y \in E\} = \ker(f)$.

2. $\{0_E\}^\perp = \{x \in E : f(x,0) = 0\} = E$.

Definition 2.4.3 *An invertible square matrix A is said to be* ***orthogonal*** *if $A^t = A^{-1}$.*

Clearly, a sufficient and necessary condition for A to be orthogonal is that $AA^t = A^tA = I$, where I is the identity matrix.

Example 2.4.2 *By the above definition, the following matrices*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \textit{with } \theta \in \mathbb{R},$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

are orthogonal.

Proposition 2.4.1 *Let $A \in M_n(\mathbb{R})$ be an orthogonal matrix. Then* $\det(A) = \pm 1$.

Proof. Since $A^t = A^{-1}$, we conclude that $A^t A = I_n$. Taking determinants gives

$$\det(A^t A) = \det(A^t)\det(A) = (\det A)^2 = \det(I_n) = 1.$$

Hence $\det(A) = \pm 1$. ∎

Lemma 2.4.1 *For each matrix $A \in M_n(\mathbb{K})$ and for each $x \in \mathbb{K}^n$, we have the inequality*

$$\|Ax\| \leq \|A\|\,\|x\|.$$

Proof. Recall the induced operator norm:

$$\|A\| = \sup_{y \in \mathbb{K}^n \setminus \{0\}} \frac{\|Ay\|}{\|y\|}.$$

If $x = 0$ the inequality is trivial. If $x \neq 0$, substituting $y = x$ in the definition gives

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|.$$

Multiplying both sides by $\|x\|$ yields $\|Ax\| \leq \|A\|\,\|x\|$, which proves the lemma. ∎

2.4.1 Orthogonal bases

Definition 2.4.4 *An orthogonal basis of a finite-dimensional vector space E equipped with a symmetric bilinear form f is a basis $B = \{e_1, \dots, e_n\}$ such that for all $i \neq j$,*

$$f(e_i, e_j) = 0.$$

Proprieties 2.4.1

- *If B is orthogonal and $f(e_i, e_i) \neq 0$ for all i, then B is linearly independent.*
- *An orthonormal basis satisfies additionally $f(e_i, e_i) = 1$ for all i.*
- *Orthogonalization exists for inner-product spaces (Gram-Schmidt). For general symmetric bilinear forms, diagonalization yields an orthogonal basis when the field and form permit.*

Coordinate criterion. If $A = (a_{ij})$ is the matrix of f in B, then B is orthogonal iff $a_{ij} = 0$ for all $i \neq j$.

Example 2.4.3

$$f((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2,$$

the canonical basis $\{(1, 0), (0, 1)\}$ is orthogonal and orthonormal.

2.4.2 Matrix norms

Definition 2.4.5 *Let E be a vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A* matrix norm *over E is a map $\|\cdot\|$ assigning to each $m \times n$ matrix A a nonnegative real number such that for all matrices A, B of compatible sizes and all scalars α:*

1. $\|A\| \geq 0, \quad \|A\| = 0 \iff A = 0,$
2. $\|\alpha A\| = |\alpha| \, \|A\|,$
3. $\|A + B\| \leq \|A\| + \|B\|.$

In this case, the couple $(E, \|.\|)$ is called normed vector space or normed space. So, a normed space E is a v. space endowed by a norm.

Example 2.4.4 *For* $A = (a_{ij}) \in M_n(K)$ *define*

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|,$$

$$\|A\|_2 = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}, \qquad \|A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p\right)^{1/p} \qquad (1 \le p < \infty).$$

- *For* $x = (-1, 1, -2)^t$*, we have*

$$\|x\|_1 = |1|+|1|+|2| = 4, \qquad \|x\|_2 = \sqrt{1^2+1^2+2^2} = \sqrt{6}, \qquad \|x\|_\infty = \max(|1|, |1|, |2|) = 2.$$

- *For* $A = \begin{pmatrix} -1 & -2 \\ 7 & 3 \end{pmatrix} \in M_2(\mathbb{R})$*, we have*

  Column sums: $c_1 = |1| + |7| = 8$*,* $c_2 = |2| + |3| = 5$*.*

  Row sums: $r_1 = |1| + |2| = 3$*,* $r_2 = |7| + |3| = 10$*. Therefore*

$$\|A\|_1 = \max(c_1, c_2) = 8, \qquad \|A\|_\infty = \max(r_1, r_2) = 10,$$

$$\|A\|_2 = \sqrt{1^2+2^2+7^2+3^2} = \sqrt{63} = 3\sqrt{7}.$$

2.4.3 Scalar Product (Inner product)

Definition 2.4.6 (Scalar product) *Let* E *be a real vector space. A* scalar product *(inner product) on* E *is a map*

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$$

satisfying for all $u, v, w \in E$ *and all* $\alpha \in \mathbb{R}$*:*

1. ***Linearity in the first argument:*** $\langle \alpha u + v, w\rangle = \alpha\langle u, w\rangle + \langle v, w\rangle$.
2. ***Symmetry:*** $\langle u, v\rangle = \langle v, u\rangle$.
3. ***Positive-definiteness:*** $\langle v, v\rangle \ge 0$ *with equality iff* $v = 0$.

A pair $(E, \langle \cdot, \cdot \rangle)$ *is called a real inner product space.*

Remark 2.4.1 *The scalar product induces a norm* $\|\cdot\|$ *on* E *defined by*

$$\|v\| = \sqrt{\langle v, v\rangle} \quad (v \in E).$$

The distance between $u, v \in E$ *is* $\|u - v\|$*. The induced norm satisfies the triangle inequality and is related to the inner product by the polarization identity:*

$$\langle u, v\rangle = \tfrac{1}{2}\Big(\|u+v\|^2 - \|u\|^2 - \|v\|^2\Big) \qquad (u, v \in E).$$

Proposition 2.4.2 (Cauchy–Schwarz Inequality) *For all* $u, v \in E$*,*

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|.$$

Equality holds iff u *and* v *are linearly dependent.*

Proof. Assume $v \neq 0$. Consider the function

$$f(t) = \|u - tv\|^2 = \langle u - tv, u - tv\rangle.$$

Expanding:

$$f(t) = \|u\|^2 - 2t\langle u, v\rangle + t^2\|v\|^2.$$

Since $f(t) \geq 0$ for all $t \in \mathbb{R}$, the discriminant of this quadratic satisfies:

$$\Delta = (-2\langle u, v\rangle)^2 - 4\|v\|^2\|u\|^2 \leq 0.$$

Thus,

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

If $v = 0$, then both sides are zero and the inequality holds trivially. ∎

Example 2.4.5 (Standard dot product:) *On* $\mathbb{R}^n$*,*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x^{\mathrm{t}} y,$$

with induced norm $\|x\| = \sqrt{x^{\mathrm{t}}x}$*. In particular, for*

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, we have \langle x, y\rangle = \langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2.$$

Example 2.4.6 (Function space:) *On* $C([a,b])$*,*

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t,$$

with induced norm $\|f\| = \left(\int_a^b f(t)^2\,\mathrm{d}t\right)^{1/2}$.

Example 2.4.7 (Frobenius Inner Product on Matrices:) *Let* $A, B \in M_n(\mathbb{R})$*. Define the inner product:*

$$\langle A, B\rangle = \mathrm{tr}(A^{\mathsf{t}}B)$$

This satisfies the properties of an inner product:

- ***Linearity:*** $\langle \alpha A + C, B\rangle = \alpha\langle A, B\rangle + \langle C, B\rangle$
- ***Symmetry:*** $\langle A, B\rangle = \langle B, A\rangle$
- ***Positive-definiteness:*** $\langle A, A\rangle = \mathrm{tr}(A^{\mathsf{t}}A) \geq 0$*, with equality iff* $A = 0$

This is called the Frobenius inner product *on* $M_n(\mathbb{R})$.

Theorem 2.4.1 (Equivalent Conditions for Orthogonality:) *Let* $A \in M_n(\mathbb{R})$*. The following statements are equivalent:*

(i) *A is orthogonal:* $A^{\mathsf{t}}A = I_n$

(ii) *For all* $x \in \mathbb{R}^n$, $\|Ax\| = \|x\|$

(iii) *For all* $x, y \in \mathbb{R}^n$, $\langle Ax, Ay\rangle = \langle x, y\rangle$

Proof.

$(i) \Rightarrow (ii)$ Assume $A^{\mathsf{t}}A = I_n$. Then:

$$\|Ax\|^2 = \langle Ax, Ax\rangle = x^{\mathsf{t}}A^{\mathsf{t}}Ax = x^{\mathsf{t}}x = \|x\|^2$$

So $\|Ax\| = \|x\|$.

$(ii) \Rightarrow (iii)$ Assume $\|Ax\| = \|x\|$. Then:

$$\|A(x+y)\|^2 = \|x+y\|^2$$

Expanding both sides:

$$\|Ax\|^2 + 2\langle Ax, Ay\rangle + \|Ay\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

Thus:

$$\langle Ax, Ay\rangle = \langle x, y\rangle$$

$(iii) \Rightarrow (i)$ Assume $\langle Ax, Ay\rangle = \langle x, y\rangle$. Then:

$$x^{\mathsf{t}}(A^{\mathsf{t}}A - I_n)y = 0 \quad \text{for all } x, y$$

So $A^{\mathsf{t}}A = I_n$, hence A is

■

2.5 Exercises

In this section, we present a series of application-oriented exercises designed to illustrate the practical relevance of the main theoretical results established earlier.

2.5.1 Exercises Statements

Exercise 2.5.1 *Determine whether the following expressions are bilinear forms, where* $u = (x_1, x_2)$ *et* $v = (y_1, y_2)$*:*
$f_1(u,v) = 2x_1y_2 - 3x_2y_1$, $f_2(u,v) = x_1x_2 + 3y_1y_2$, $f_3(u,v) = x_1 + y_2$.

Exercise 2.5.2 *Show that the determinant of the square matrix:* $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ *is a bilinear form of its two column vectors. Show that it is alternating. What is its matrix representation relative to the canonical basis?*

Exercise 2.5.3 *Let* $\varphi : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \longrightarrow \mathbb{R}$*, defined by* $\varpi(A,B) = \mathrm{Tr}({}^tAB)$*. Verify that* φ *is a bilinear map. What is its matrix in the canonical basis of* $M_2(\mathbb{R})$*?*

Exercise 2.5.4 *Determine the bilinear form on* $\mathbb{R}^3$ *whose matrix in the canonical basis is:*

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Exercise 2.5.5 *Consider the two bilinear forms:*

- $f(x,y) = x_1y_2 + x_2(-y_1 + y_3) - x_3y_2$, *sur* $E = \mathbb{R}^3$.
- $\varphi(x,y) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$, *sur* $E = \mathbb{R}^4$.

1. *Explicitly write the matrices of these bilinear forms.*
2. *Compute the rank of each.*

Exercise 2.5.6 *Let $E = \mathbb{R}^3$. Define the bilinear form f on E (in the canonical basis $B = \{e_1, e_2, e_3\}$) by:*

$$f(x, y) = x_1y_1 + 2x_2y_2 + 3x_3y_3 - x_1y_2 - x_2y_1 - x_1y_3 - x_3y_1$$

1. *Verify that f is symmetric. Give the matrix of f in the basis B.*
2. *Let $\tilde{e}_1 = e_1, \tilde{e}_2 = e_1 + e_2, \tilde{e}_3 = 2e_1 + e_2 + 3e_3$. Show that $\tilde{B} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a basis of E.*
3. *What is the matrix of f in the basis $\tilde{B}$? Deduce that f is positive definite. What can be concluded?*

Exercise 2.5.7 *Consider the bilinear form defined on $\mathbb{R}^2$ by:*

$$f((x_1, x_2), (y_1, y_2)) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

1. *What is its matrix A in the basis $\{u_1 = (1, 0), u_2 = (1, 1)\}$?*
2. *What is its matrix B in the basis $\{v_1 = (2, 1), v_2 = (1, -1)\}$?*
3. *What is the change-of-basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$?*
4. *Verify the relation $B = P^tAP$.*

Exercise 2.5.8 *Let φ and ψ be two linear forms on a vector space V and let $f : V \times V \longrightarrow K$ and define by:*

$$f(v_1, v_2) = \varphi(v_1).\psi(v_2).$$

1. *Show that f is a bilinear form.*
2. *Suppose $V = \mathbb{R}^2$, and let $\varphi(v_1) = 2x_1 - 5y_1$, $\psi(v_2) = -3x_2 + 4y_2$ be two linear forms defined on V.*
   - *(a) Write the expression of the bilinear form f.*
   - *(b) Give a matrix representation for f.*
   - *(c) What is its rank?*
   - *(d) What is its kernel?*

2.5.2 Exercises Corrections

Solution 2.5.1 *Let $u = (x_1, x_2),\ v = (y_1, y_2) \in \mathbb{R}^2$. Determine whether the following expressions define bilinear forms:*

(1) $f_1(u, v) = 2x_1y_2 - 3x_2y_1$

This expression is bilinear because it is a linear combination of products of coordinates from u and v. Specifically:

$$f_1(u, v) = 2x_1y_2 - 3x_2y_1$$

is linear in u when v is fixed, and linear in v when u is fixed.

Thus, f_1 is a bilinear form.

(2) $f_2(u, v) = x_1x_2 + 3y_1y_2$

This expression is not bilinear because it involves products of coordinates within the same vector:

$$x_1x_2 \quad \text{(depends only on } u), \qquad y_1y_2 \quad \text{(depends only on } v)$$

It is not linear in v when u is fixed, nor in u when v is fixed.

Thus, f_2 is not a bilinear form.

(3) $f_3(u, v) = x_1 + y_2$

This expression is additive but not bilinear. It is linear in each variable separately, but not multilinear:

$$f_3(u, v) = x_1 + y_2$$

There are no products between coordinates of u and v.

Thus, f_3 is not a bilinear form.

(4) $f_4(u, v) = 1$ *is not a bilinear form.*

(5) $f_5(u, v) = 0$ *it is bilinear form.*

Solution 2.5.2 *Let:*

$$M = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

The determinant of M is:

$$\det(M) = x_1y_2 - x_2y_1$$

Bilinearity:

Let $u = (x_1, x_2),\ v = (y_1, y_2)$*. Then:*

$$\det(u, v) = x_1 y_2 - x_2 y_1$$

This is bilinear because it is linear in each argument when the other is fixed.

Alternation:

We check whether $\det(u, u) = 0$ *for all* $u \in \mathbb{R}^2$*:*

$$\det(u, u) = x_1 x_2 - x_2 x_1 = 0$$

Also, swapping u *and* v *changes the sign:*

$$\det(v, u) = y_1 x_2 - y_2 x_1 = -\det(u, v)$$

Thus, the determinant is an alternating bilinear form.

Matrix Representation:

Let $B = \{e_1, e_2\}$ *be the canonical basis of* $\mathbb{R}^2$*. Then the matrix of the bilinear form* $\det(u, v)$ *is:*

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This satisfies:

$$\det(u, v) = u^t M v$$

Solution 2.5.3 *Let* $\varphi : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$ *be defined by:*

$$\varphi(A, B) = \mathrm{Tr}(A^t B)$$

Bilinearity:

$$\varphi(A, B) = \mathrm{Tr}(A^t B)$$

is bilinear because:

- $A \mapsto A^t B$ *is linear.*

- $B \mapsto A^t B$ *is linear.*

- *The trace is a linear map on matrices.*

Thus, φ is bilinear.

Matrix Representation:

Let $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the canonical basis of $M_2(\mathbb{R})$, where E_{ij} is the matrix with 1 at position (i, j) and 0 elsewhere.
We compute $\varphi(E_{ij}, E_{kl}) = \mathrm{Tr}(E_{ij}^t E_{kl})$. Note:

$$E_{ij}^t = E_{ji}, \quad E_{ji}E_{kl} = \delta_{ik}E_{jl} \Rightarrow \mathrm{Tr}(E_{ji}E_{kl}) = \delta_{ik}\delta_{jl}$$

So the matrix of φ in the basis B is the 4×4 identity matrix:

$$M = I_4$$

Thus, the matrix of φ in the canonical basis of $M_2(\mathbb{R})$ is the identity matrix.

Solution 2.5.4 *Let $b(x, y)$ be the bilinear form defined by:*

$$b(x,y) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^t \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Expanding the expression, we get:

$$\begin{aligned} b(x,y) &= x_1(1y_1 + 2y_2 + 3y_3) + x_2(2y_1 + 3y_2 + 4y_3) + x_3(3y_1 + 4y_2 + 5y_3) \\ &= x_1y_1 + 2x_1y_2 + 3x_1y_3 + 2x_2y_1 + 3x_2y_2 + 4x_2y_3 + 3x_3y_1 + 4x_3y_2 + 5x_3y_3. \end{aligned}$$

Grouping terms by x_i, we obtain:

$$b(x,y) = x_1y_1 + 4x_1y_2 + 6x_1y_3 + 3x_2y_2 + 8x_2y_3 + 5x_3y_3.$$

Solution 2.5.5 *1. To compute the elements of the matrix $M_f = (a_{ij})$ associated with the bilinear form f, one can apply the relation:*

$$a_{ij} = f(e_i, e_j),$$

or proceed directly as follows.

Let $x = (x_1, x_2, x_3)^t$, $y = (y_1, y_2, y_3)^t$. *Then:*

$$\begin{aligned} f(x,y) &= x_1y_2 + x_2(-y_1 + y_3) + x_3y_2 \\ &= x_1(0y_1 + y_2 + 0y_3) + x_2(-y_1 + 0y_2 + y_3) + x_3(0y_1 - y_2 + 0y_3) \\ &= (x_1, x_2, x_3)^t \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \end{aligned}$$

Thus, the matrix of f *is:*

$$M_f = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

- ***The rank*** *of* f *is equal to the rank of its matrix. Since:*

  $$\det(M_f) = 0,$$

  we have $\operatorname{rank}(M_f) < 3$.

  However, the 2×2 *submatrix*

  $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  has determinant $-1 \neq 0$, *so its rank is 2.*

  Therefore, the bilinear form f *has rank 2.*

2. *Following the same approach, the matrix* M' *associated with the bilinear form* φ *is given by:*

Let $x = (x_1, x_2, x_3, x_4)^t$, $y = (y_1, y_2, y_3, y_4)^t$. *Then:*

$$\begin{aligned}\varphi(x,y) &= x_1y_2 - y_1x_2 + x_3y_4 - y_3x_4\\ &= x_1(0y_1 + y_2 + 0y_3 + 0y_4) + x_2(-y_1 + 0y_2 + 0y_3 + 0y_4)\\ &\quad + x_3(0y_1 + 0y_2 + 0y_3 + y_4) + x_4(0y_1 + 0y_2 - y_3 + 0y_4)\\ &= (x_1, x_2, x_3, x_4)^t \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.\end{aligned}$$

Thus, the matrix of φ *is:*

$$M' = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.$$

We compute the determinant:

$$\det(M') = 1 \neq 0 \Rightarrow \operatorname{rank}(M') = 4.$$

Remark 2.5.1 *By applying Gaussian elimination (swapping rows* $L_1 \leftrightarrow L_2$ *and* $L_3 \leftrightarrow L_4$*), we obtain the row-echelon matrix:*

$$\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

which contains no zero rows. Therefore, the rank is 4.

Solution 2.5.6 *1. The matrix of the bilinear form* f *in the canonical basis is:*

$$M_f = \begin{pmatrix} 1 & -1 & -1\\ -1 & 2 & 0\\ -1 & 0 & 3\end{pmatrix}.$$

Since M_f *is symmetric, the bilinear form* f *is symmetric.*

2. *The family* $B' = \{e'_1, e'_2, e'_3\}$ *consists of 3 vectors. To show that it is a basis of* E*, it suffices to prove that it is linearly independent*

   Let M *be the matrix whose columns are the coordinates of the vectors* e'_i *in the canonical basis:*

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

   We compute:

$$\det(M) = 1 \neq 0 \Rightarrow B' \text{ is a basis of } E.$$

3. *The change-of-basis matrix from* B *to* B' *is* $P = M$*. Therefore, the matrix of the bilinear form* f *in the basis* B' *is:*

$$M'_f = P^t M_f P.$$

   Computing step-by-step:

$$P^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

   After performing the multiplication:

$$M'_f = P^t M_f P = I_3.$$

   Therefore, in the basis B'*, the bilinear form* f *is represented by the identity matrix. For any vectors*

$$x = x_1 e'_1 + x_2 e'_2 + x_3 e'_3, \quad y = y_1 e'_1 + y_2 e'_2 + y_3 e'_3,$$

   with coordinate vectors:

$$X' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad Y' = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

   we have:

$$f(x, y) = X'^t I_3 Y' = \langle X_0, Y_0 \rangle.$$

   Thus, f *defines a scalar product on* E*, and is therefore a positive definite bilinear form.*

Solution 2.5.7 ***Matrix of f in the canonical basis***

Let $e_1 = (1,0), e_2 = (0,1)$. We compute the matrix M of f in this basis using.

$$f(u,v) = u^t M v$$

Compute entries $m_{ij} = f(e_i, e_j)$:

$$f(e_1,e_1) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 \cdot 0 + 0 \cdot 0 = 2$$
$$f(e_1,e_2) = 2 \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 1 + 0 \cdot 1 = -3$$
$$f(e_2,e_1) = 2 \cdot 0 \cdot 1 - 3 \cdot 0 \cdot 0 + 1 \cdot 1 = 1$$
$$f(e_2,e_2) = 2 \cdot 0 \cdot 0 - 3 \cdot 0 \cdot 1 + 1 \cdot 1 = 1$$

So the matrix in the canonical basis is:

$$M = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$$

1. ***Matrix A in basis $\mathcal{U} = \{u_1, u_2\}$***

   We compute $A_{ij} = f(u_i, u_j)$:

$$f(u_1,u_1) = f((1,0),(1,0)) = 2$$
$$f(u_1,u_2) = f((1,0),(1,1)) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 \cdot 1 + 0 \cdot 1 = -1$$
$$f(u_2,u_1) = f((1,1),(1,0)) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 \cdot 0 + 1 \cdot 0 = 2$$
$$f(u_2,u_2) = f((1,1),(1,1)) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 \cdot 1 + 1 \cdot 1 = 0$$

   So:

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

2. ***Matrix B in basis $\mathcal{V} = \{v_1, v_2\}$***

   Compute $B_{ij} = f(v_i, v_j)$:

$$f(v_1,v_1) = f((2,1),(2,1)) = 2 \cdot 2 \cdot 2 - 3 \cdot 2 \cdot 1 + 1 \cdot 1 = 8 - 6 + 1 = 3$$
$$f(v_1,v_2) = f((2,1),(1,-1)) = 2 \cdot 2 \cdot 1 - 3 \cdot 2 \cdot (-1) + 1 \cdot (-1) = 4 + 6 - 1 = 9$$
$$f(v_2,v_1) = f((1,-1),(2,1)) = 2 \cdot 1 \cdot 2 - 3 \cdot 1 \cdot 1 + (-1) \cdot 1 = 4 - 3 - 1 = 0$$
$$f(v_2,v_2) = f((1,-1),(1,-1)) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 \cdot (-1) + (-1) \cdot (-1) = 2 + 3 + 1 = 6$$

So:

$$B = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$$

3. ***Change-of-basis matrix*** P ***from*** $\mathcal{U}$ ***to*** $\mathcal{V}$

   We express each v_i *in terms of* $u_1 = (1,0), u_2 = (1,1)$. *Let's solve:*

$$v_1 = a_1u_1 + a_2u_2 = a_1(1,0) + a_2(1,1) = (a_1 + a_2, a_2) \Rightarrow \begin{cases} a_1 + a_2 = 2 \\ a_2 = 1 \end{cases} \Rightarrow a_1 = 1$$

   So $v_1 = u_1 + u_2$

   Similarly:

$$v_2 = b_1u_1 + b_2u_2 = b_1(1,0) + b_2(1,1) = (b_1 + b_2, b_2) \Rightarrow \begin{cases} b_1 + b_2 = 1 \\ b_2 = -1 \end{cases} \Rightarrow b_1 = 2$$

   So $v_2 = 2u_1 - u_2$ *Hence:*

$$P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

4. ***Verify*** $B = P^tAP$

   Compute:

$$P^t = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

   First compute P^tA*:*

$$P^\top A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix}$$

   Then compute P^tAP*:*

$$\begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix} = B.$$

   Verified: $B = P^tAP$

Solution 2.5.8 *1. Let* $v_1, v_1', v_2, v_2' \in V$, *and* $\lambda \in K$. *We verify linearity in each argument:*

Linearity in the first variable:

$$f(v_1 + v_1', v_2) = \varphi(v_1 + v_1') \cdot \psi(v_2) = (\varphi(v_1) + \varphi(v_1')) \cdot \psi(v_2) = f(v_1, v_2) + f(v_1', v_2)$$

$$f(\lambda v_1, v_2) = \varphi(\lambda v_1) \cdot \psi(v_2) = \lambda \varphi(v_1) \cdot \psi(v_2) = \lambda f(v_1, v_2)$$

Linearity in the second variable:

$$f(v_1, v_2 + v_2') = \varphi(v_1) \cdot \psi(v_2 + v_2') = \varphi(v_1) \cdot (\psi(v_2) + \psi(v_2')) = f(v_1, v_2) + f(v_1, v_2')$$

$$f(v_1, \lambda v_2) = \varphi(v_1) \cdot \psi(\lambda v_2) = \varphi(v_1) \cdot \lambda \psi(v_2) = \lambda f(v_1, v_2)$$

Thus, f is bilinear.

2. *Suppose $V = \mathbb{R}^2$, with*

$$\varphi(v_1) = 2x_1 - 5y_1, \quad \psi(v_2) = -3x_2 + 4y_2.$$

Let $v_1 = (x_1, y_1),\ v_2 = (x_2, y_2) \in \mathbb{R}^2$.

(a) ***Expression of the bilinear form*** f

$$f(v_1, v_2) = \varphi(v_1) \cdot \psi(v_2) = (2x_1 - 5y_1)(-3x_2 + 4y_2)$$

Expand the product:

$$f(v_1, v_2) = -6x_1x_2 + 8x_1y_2 + 15y_1x_2 - 20y_1y_2$$

(b) ***Matrix representation of*** f *We seek a matrix $M \in \mathbb{R}^{2\times 2}$ such that:*

$$f(v_1, v_2) = v_1^\top M v_2$$

From the expression:

$$f(v_1, v_2) = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 15 & -20 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

So the matrix of f is:

$$M = \begin{bmatrix} -6 & 8 \\ 15 & -20 \end{bmatrix}$$

(c) ***Rank of*** *f Since $f(v_1, v_2) = \varphi(v_1) \cdot \psi(v_2)$, it is the tensor product of two linear forms. Therefore, the matrix M is of rank 1.*

Thus,$\operatorname{rank}(f) = 1$.

(d) ***Kernel of*** *f The kernel of a bilinear form f is the set:*

$$\ker(f) = \{v \in V \mid f(v, w) = 0 \textit{ for all } w \in V\}$$

We have:

$$f(v_1, v_2) = \varphi(v_1) \cdot \psi(v_2) \Rightarrow f(v_1, v_2) = 0 \textit{ for all } v_2 \iff \varphi(v_1) = 0$$

So:

$$\ker(f) = \ker(\varphi) = \{(x, y) \in \mathbb{R}^2 \mid 2x - 5y = 0\} \Rightarrow \ker(f) = \textit{span}\,((5, 2))$$

Thus, the kernel of f is the line $2x - 5y = 0$, i.e., the subspace generated by $(5, 2)$.

Conclusion

In this chapter, we have developed a comprehensive understanding of bilinear forms and their role in the study of vector spaces. By examining their algebraic properties and matrix representations, we have seen how bilinear forms provide a powerful framework for analyzing vector interactions, defining orthogonality, and constructing quadratic forms. The connection between bilinear forms and linear transformations further highlights their versatility in both theoretical and applied contexts.

Whether used to define inner products, study symmetric matrices, or explore geometric structures, bilinear forms serve as a bridge between abstract linear theory and concrete computational techniques. Mastery of this topic equips students with essential tools for advanced studies in linear algebra, differential geometry, and functional analysis.

Chapter 3

Symmetric bilinear forms and quadratic forms

Introduction

Quadratic forms and bilinear forms are important tools in mathematics. They appear in many areas like geometry, optimization, statistics, and physics. In this chapter, we focus on their basic algebraic properties.

We start by defining symmetric and antisymmetric bilinear forms, and how they relate to concepts like orthogonality and isotropic vectors. Then we show how every quadratic form is linked to a symmetric bilinear form, which helps us study its rank, kernel, and definiteness.

Finally, we learn how to simplify quadratic forms using the Gauss method and how to classify them using their signature. These ideas help us understand the shape and behavior of quadratic forms in different mathematical settings see [3, 2, 7, 8].

3.1 Isotropic Cone

Let E be a vector space over a field $\mathbb{K}$, and let $b \in \mathcal{B}(E, \mathbb{K})$ (the space of bilinear forms on E).

Definition 3.1.1 *(i) b is* ***symmetric*** *if for all $(x, y) \in E^2$, we have $b(x, y) = b(y, x)$.*

(ii) b is ***antisymmetric*** *if for all $(x, y) \in E^2$, we have $b(x, y) = -b(y, x)$.*

A form $b \in \mathcal{B}(E, \mathbb{K})$ is antisymmetric if and only if $b(x, x) = 0$ for all $x \in E$.

We saw in the previous chapter that the dot product on $\mathbb{R}^n$ is a symmetric bilinear form. An example of an antisymmetric bilinear form is the determinant of two column vectors X and Y in $\mathbb{K}^2$, defined by:

$$\det(X, Y) = x_1 y_2 - x_2 y_1$$

Definition 3.1.2 (Orthogonality or Conjugacy) *Let b be a symmetric bilinear form on E. Two vectors x and y in E are said to be orthogonal or conjugate with respect to b if $b(x, y) = 0$. A family $\{e_i\}_{i \in I}$ of vectors in E is said to be b-conjugate if for all $i \neq j$, $b(e_i, e_j) = 0$.*

Definition 3.1.3 (Orthogonal Complement) *Let F be a subspace of E and $b \in \mathcal{B}(E, \mathbb{K})$. The orthogonal complement of F with respect to b is:*

$$F^{\perp} = \{x \in E \mid \forall y \in F,\ b(x, y) = 0\}$$

Definition 3.1.4 (Isotropic Vectors and Isotropic Cone) *Let E be a vector space over $\mathbb{K}$, and let b be a symmetric bilinear form on E. A vector $x \in E$ is said to be isotropic for b if $b(x, x) = 0$. The set of isotropic vectors for b is called the* ***isotropic cone****. The form b is said to be* ***definite*** *if its isotropic cone is reduced to the zero vector.*

Remark 3.1.1 *Every vector in* $\ker b$ *is isotropic. If b is definite, then it is non-degenerate. The converse is not necessarily true.*

3.2 Quadratic Forms

A quadratic form on a vector space E is a function that, to a vector $x = (x_1, \ldots, x_n) \in E$, associates a homogeneous polynomial of degree 2 in the variables x_i. This is the original definition of a quadratic form. It can also be defined as associated with a symmetric bilinear form.

Definition 3.2.1 *Let b be a symmetric bilinear form on E. The quadratic form associated with b is the function:*

$$q : E \to \mathbb{K}, \quad x \mapsto q(x) = b(x, x)$$

Examples 3.2.1 *1. Consider the symmetric bilinear form*

$$\varphi : \mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathbb{R}, \quad (f,g) \longmapsto \int_0^1 f(t)g(t)\,dt.$$

The quadratic form associated with φ is:

$$q_\varphi : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathbb{R}, \quad f \longmapsto q_\varphi(f) = \varphi(f,f) = \int_0^1 f(t)^2\,dt.$$

2. Consider the symmetric bilinear form

$$f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad ((x_1,x_2),(y_1,y_2)) \longmapsto x_1y_2 - x_2y_1.$$

The quadratic form associated with f is:

$$q_f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x_1,x_2) \longmapsto q_f(x_1,x_2) = f((x_1,x_2),(x_1,x_2)) = x_1x_2 - x_2x_1 = 0.$$

Note: Every antisymmetric bilinear form yields a zero quadratic form.

Proprieties 3.2.1 *Let $f : E \times E \to \mathbb{K}$ be a symmetric bilinear form over a vector space E, and let $q(x) = f(x,x)$ be the associated quadratic form. Then the following identities hold:*

1. $q(ax) = a^2 q(x), \quad \forall x \in E, \forall a \in \mathbb{K}.$
2. $q(x+y) = q(x) + q(y) + 2f(x,y), \quad \forall x,y \in E$ *(Polarization Identity)*
3. $q(x+y) + q(x-y) = 2(q(x)+q(y)), \quad \forall x,y \in E$ *(Median Identity)*

Proof.

1. Let $x \in E$ and $a \in \mathbb{K}$. Then:

$$q(ax) = f(ax,ax) = aaf(x,x) = a^2 q(x).$$

2. Let $x, y \in E$. Then:

$$\begin{aligned} q(x+y) &= f(x+y,x+y) \\ &= f(x,x) + f(x,y) + f(y,x) + f(y,y) \\ &= q(x) + 2f(x,y) + q(y). \end{aligned}$$

3. From the polarization identity, we have:

$$q(x+y) = q(x) + q(y) + 2f(x,y), \quad q(x-y) = q(x) + q(y) - 2f(x,y).$$

Adding both expressions:

$$q(x+y) + q(x-y) = 2(q(x) + q(y)).$$

■

Examples 3.2.2 *1. On $E = \mathbb{K}$, the simplest quadratic form is $x \mapsto x^2$.*

2. The square of a linear form $\varphi \in E^*$ is a quadratic form:

$$x \mapsto (\varphi(x))^2, \quad \textit{polar form: } b(x,y) = \varphi(x)\varphi(y)$$

3. The product of two linear forms $\varphi_1, \varphi_2 \in E^*$ is a quadratic form:

$$x \mapsto \varphi_1(x)\varphi_2(x), \quad \textit{polar form: } b(x,y) = \varphi_1(x)\varphi_2(y)$$

4. The form $A \mapsto Tr(A^2)$ defined from $\mathcal{M}_n(\mathbb{K})$ to $\mathbb{K}$ is a quadratic form.

Theorem 3.2.1 *Let q be a quadratic form on E. Then there exists a unique symmetric bilinear form b on E such that $b(x,x) = q(x)$, called the* ***polar form*** *of q, and given by:*

$$b(x,y) = \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right]$$

Example 3.2.1 *Let the quadratic form*

$$q : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto q(x_1, x_2) = x_1^2 + 2x_2^2 - 3x_1x_2.$$

The associated polar (bilinear) form f is defined by:

$$f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad ((x_1, x_2), (y_1, y_2)) \longmapsto f((x_1, x_2), (y_1, y_2)),$$

with

$$f((x_1, x_2), (y_1, y_2)) = \frac{1}{2}\left[q((x_1 + y_1, x_2 + y_2)) - q((x_1, x_2)) - q((y_1, y_2))\right].$$

Explicitly, we compute:

$$\begin{aligned} q(x_1+y_1, x_2+y_2) &= (x_1+y_1)^2 + 2(x_2+y_2)^2 - 3(x_1+y_1)(x_2+y_2) \\ &= x_1^2 + 2x_2^2 - 3x_1x_2 + y_1^2 + 2y_2^2 - 3y_1y_2 \\ &\quad + 2x_1y_1 + 4x_2y_2 - 3x_1y_2 - 3x_2y_1 - 3y_1x_2 - 3y_2x_1. \end{aligned}$$

Subtracting $q(x_1, x_2) + q(y_1, y_2)$*, we obtain:*

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 + 2x_2y_2 - \frac{3}{2}x_1y_2 - \frac{3}{2}x_2y_1.$$

3.2.1 Matrix Associated with a Quadratic Form

Definition 3.2.2 *Let* q *be a quadratic form on a finite-dimensional vector space* E *over a field* $\mathbb{K}$*, and let* B *be a basis of* E*. The matrix associated with* q *relative to the basis* B*, denoted* $M_B(q)$*, is the matrix associated with the polar form* f *of* q *with respect to* B*, that is:*

$$M_B(q) = M_B(f).$$

Remark 3.2.1 *The matrix associated with a quadratic form is symmetric, i.e.,*

$$M_B(q) = M_B(q)^t.$$

Definition 3.2.3 *Let* q *be a quadratic form on* $\mathbb{R}^n$*, and let* $B = \{e_1, e_2, \dots, e_n\}$ *be the canonical basis. The matrix associated with* q *relative to the basis* B *is given by:*

$$M_B(q) = \frac{1}{2}\begin{pmatrix} \frac{\partial q}{\partial x_1}(e_1) & \frac{\partial q}{\partial x_2}(e_1) & \cdots & \frac{\partial q}{\partial x_n}(e_1) \\ \frac{\partial q}{\partial x_1}(e_2) & \frac{\partial q}{\partial x_2}(e_2) & \cdots & \frac{\partial q}{\partial x_n}(e_2) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q}{\partial x_1}(e_n) & \frac{\partial q}{\partial x_2}(e_n) & \cdots & \frac{\partial q}{\partial x_n}(e_n) \end{pmatrix}.$$

Example 3.2.2 *Let the quadratic form*

$$q : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (x_1, x_2, x_3) \longmapsto q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3.$$

We compute the partial derivatives:

$$\frac{\partial q}{\partial x_1} = 2x_1 + 2x_2 - 4x_3, \quad \frac{\partial q}{\partial x_2} = 2x_1 + 6x_2, \quad \frac{\partial q}{\partial x_3} = -4x_1 + 6x_3.$$

Therefore, the matrix associated with q relative to the canonical basis of $\mathbb{R}^3$ is:

$$M_B(q) = \frac{1}{2}\begin{pmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -4 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 3 \end{pmatrix}.$$

3.2.2 Quadratic Form Associated with a Square Matrix

Proposition 3.2.1 *Let $M = (a_{ij})_{1\leq i<j\leq n}$ be a symmetric matrix associated with a quadratic form q relative to the canonical basis $\{e_1, e_2, \dots, e_n\}$ of a vector space E. Then for any vector*

$$x = \sum_{i=1}^{n} x_i e_i \in E,$$

the quadratic form $q(x)$ is given by:

$$q(x) = \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{1\leq i<j\leq n} a_{ij}x_ix_j.$$

Proof. Let $M = (a_{ij})$ be a symmetric matrix, i.e., $a_{ij} = a_{ji}$ or equivalently $M^t = M$. Then:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let f be the polar form associated with the quadratic form q, so that:

$$q(x) = f(x, x) = x^tMx,$$

where $x = (x_1, x_2, \dots, x_n)^t \in \mathbb{R}^n$, and $M = (a_{ij})$ is a symmetric matrix:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then:

$$q(x) = x^tMx = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

This expands to:

$$q(x) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_i \right) x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Using the symmetry $a_{ij} = a_{ji}$, we can rewrite:

$$q(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j.$$

■

Remark 3.2.2 *The diagonal of the matrix M contains the coefficients of the squared terms x_i^2.*

Example 3.2.3 *Let the symmetric matrix*

$$M = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

The quadratic form q associated with M is defined by:

$$q : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (x_1, x_2, x_3) \longmapsto q(x_1, x_2, x_3),$$

with

$$q(x_1, x_2, x_3) = \sum_{i=1}^{3} a_{ii} x_i^2 + 2 \sum_{1 \leq i < j \leq 3} a_{ij} x_i x_j.$$

Substituting the entries of M, we compute:

$$\begin{aligned} q(x_1, x_2, x_3) &= x_1^2 + 2x_2^2 - x_3^2 + 2(3x_1x_2 - 2x_1x_3) \\ &= x_1^2 + 2x_2^2 - x_3^2 + 6x_1x_2 - 4x_1x_3. \end{aligned}$$

3.3 Rank, Kernel, and Isotropic Vectors of a Quadratic Form

The definitions for symmetric bilinear forms remain valid for quadratic forms due to their equivalence.

Definition 3.3.1 *The **rank**, **kernel**, and **associated matrix** of a quadratic form q are defined as the rank, kernel, and matrix of its polar form (the bilinear form associated with q).*

Definition 3.3.2 *If q is a quadratic form with real values, it is said to be* ***definite*** *if its polar form is definite, i.e.,*

$$q(x) = 0 \Rightarrow x = 0$$

In this case, q is necessarily non-degenerate.

Definition 3.3.3 *A vector $x \in E$ is called* ***isotropic*** *for the quadratic form q if $q(x) = 0$. The set of all isotropic vectors is called the* ***isotropic cone****:*

$$C(q) = \{x \in E \mid q(x) = 0\}$$

Remark 3.3.1

1. *Do not confuse the isotropic cone $C(q)$ with the kernel of q, which is defined by:*

$$\ker q = \{x \in E \mid b(x, y) = 0, \ \forall y \in E\}$$

2. *A quadratic form is non-degenerate if its kernel is reduced to the zero vector. It is definite if its isotropic cone is reduced to the zero vector.*

Example 3.3.1 *Let q be the quadratic form on $\mathbb{R}^2$ defined by:*

$$q(x) = q(x_1, x_2) = x_1^2 - x_2^2$$

Its matrix is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is non-degenerate since its kernel is reduced to $\{0\}$, but not definite because its isotropic cone is:

$$C(q) = \{x \in \mathbb{R}^2 \mid x_1^2 = x_2^2\} = \{x \in \mathbb{R}^2 \mid x_1 = \pm x_2\}$$

which is the union of two lines and not reduced to the zero vector.

3.4 Decomposition into Squares

We now study the problem of reducing quadratic forms, i.e., simplifying their expression using a suitable basis called an orthogonal basis. This amounts to finding a basis in which the matrix of a quadratic form q is diagonal, meaning that q can be expressed as a sum of squares of independent linear forms.

3.4.1 Orthogonal decomposition

Definition 3.4.1 *Let q be a quadratic form on a vector space E, and let b be its polar form. A basis $\{e_i\}$ is called:*

- ***Orthogonal*** *for q if $b(e_i, e_j) = 0$ for all $i \neq j$.*
- ***Orthonormal*** *if, in addition to being orthogonal, we have $b(e_i, e_i) = 1$ for all i.*

Remark 3.4.1 *A basis is orthogonal for a quadratic form q if its matrix in that basis is diagonal of the form:*

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

where $a_i = b(e_i, e_i)$. Then:

$$q(x) = a_1 x_1^2 + \cdots + a_n x_n^2 = \sum_{i=1}^{n} a_i x_i^2, \quad \text{with } x = \sum_{i=1}^{n} x_i e_i$$

The rank of q equals the rank of its matrix, i.e., the number of non-zero a_i.

A basis is orthonormal if the matrix associated with q is the identity. Thus:

$$q(x) = x_1^2 + \cdots + x_n^2 = \sum_{i=1}^{n} x_i^2.$$

If an orthonormal basis exists, then q has rank n and is non-degenerate. In particular, if q has real values, then an orthonormal basis exists if and only if q is positive definite.

Theorem 3.4.1 (Existence of Orthogonal Bases) *Every quadratic form q on a vector space E admits at least one orthogonal basis.*

Theorem 3.4.2 *Let q be a quadratic form on a finite-dimensional vector space E of dimension n. Then there exist linearly independent linear forms $\varphi_1, \ldots, \varphi_r \in E^*$ and non-zero constants $a_1, \ldots, a_r \in \mathbb{K}$ such that:*

$$q(x) = a_1 \varphi_1(x)^2 + \cdots + a_r \varphi_r(x)^2$$

In any such decomposition, the number r of linearly independent linear forms equals the rank of the quadratic form q.

3.4.2 Gauss Method

Let:

$$q(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j = \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{1\leq i<j\leq n} a_{ij}x_ix_j$$

be a quadratic form on K^n, with $x = (x_1, \ldots, x_n)$.

Notation 3.4.1
- *Terms of the form x_i^2 are called* ***square terms****.*
- *Terms of the form x_ix_j, with $i \neq j$, are called* ***rectangular terms****.*

The Gauss algorithm allows us to decompose q into a combination of squares of independent linear forms by recurrence, distinguishing two cases.

1. **First case:**

   q contains at least one square term. Suppose it is x_1^2 with $a_{11} = a \neq 0$. Then:

   $$q(x_1, \ldots, x_n) = ax_1^2 + x_1\varphi(x_2, \ldots, x_n) + Q(x_2, \ldots, x_n)$$

   where φ is a linear form and Q is a quadratic form in $x_2, \ldots, x_n$.

   Thus:

   $$q = a\left(x_1 + \frac{1}{2a}\varphi\right)^2 + Q - \frac{1}{4a}\varphi^2$$

   The term $x_1 + \frac{1}{2a}\varphi$ is a linear form, and:

   $$q_0(x_2, \ldots, x_n) = Q(x_2, \ldots, x_n) - \frac{1}{4a}\varphi(x_2, \ldots, x_n)^2$$

   is a quadratic form that no longer depends on x_1, to which we can apply the recurrence hypothesis.

2. **Second case:**

   q contains no square term but is not identically zero. Then it must contain at least one rectangular term. Suppose it is x_1x_2 with $a_{12} = a \neq 0$. Then:

   $$q(x_1, \ldots, x_n) = ax_1x_2 + x_1\varphi(x_3, \ldots, x_n) + x_2\psi(x_3, \ldots, x_n) + Q(x_3, \ldots, x_n)$$

   where φ and ψ are linear forms and Q is a quadratic form in $x_3, \ldots, x_n$. Then:

   $$q = a\left(x_1 + \frac{1}{a}\psi\right)\left(x_2 + \frac{1}{a}\varphi\right) + Q - \frac{1}{a}\varphi\psi$$

Let $\ell_1 = x_1 + \frac{1}{a}\psi$, $\ell_2 = x_2 + \frac{1}{a}\varphi$, and $q_0 = Q - \frac{1}{a}\varphi\psi$. Then:

$$q = a\ell_1\ell_2 + q_0 = \frac{a}{4}\left[(\ell_1+\ell_2)^2 - (\ell_1-\ell_2)^2\right] + q_0$$

Now ℓ_1, ℓ_2 are linear forms and q_0 is a quadratic form that no longer depends on x_1 or x_2, to which we can apply the recurrence hypothesis. In both cases, the Gauss algorithm allows us to decompose q into squares:

$$q(x) = a_1\varphi_1(x)^2 + \cdots + a_r\varphi_r(x)^2$$

with $a_1, \ldots, a_r \in \mathbb{K} \setminus \{0\}$ and $\varphi_1, \ldots, \varphi_r$ linearly independent linear forms on $\mathbb{K}^n$. We can complete this free family into a basis $(\varphi_1, \ldots, \varphi_r, \varphi_{r+1}, \ldots, \varphi_n)$ of $(\mathbb{K}^n)^*$. Its dual basis $(e_1, \ldots, e_n)$ of $\mathbb{K}^n$ is a basis of q-conjugate vectors (i.e., an orthogonal basis), in which the matrix of q is diagonal:

$$M = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & a_r & \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Example 3.4.1 *Let $q : \mathbb{R}^4 \to \mathbb{R}$ be the quadratic form in the canonical basis:*

$$q(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

Since q contains no square term, we choose a non-zero rectangular term, e.g., x_1x_2:

$$q(x) = x_1x_2 + x_1(x_3+x_4) + x_2(x_3+x_4) + x_3x_4$$

$$= (x_1+x_3+x_4)(x_2+x_3+x_4) - (x_3+x_4)^2 + x_3x_4$$

Using the identity:

$$\varphi_1\varphi_2 = \frac{1}{4}\left[(\varphi_1+\varphi_2)^2 - (\varphi_1-\varphi_2)^2\right]$$

we obtain:

$$q(x) = \frac{1}{4}\left[(x_1+x_2+2x_3+2x_4)^2 + (x_1-x_2)^2\right] - x_3^2 - x_4^2 - x_3x_4$$

$$= \frac{1}{4}\left[(x_1+x_2+2x_3+2x_4)^2 + (x_1-x_2)^2\right] - \left(x_3 - \frac{1}{2}x_4\right)^2 - \frac{3}{4}x_4^2$$

Example 3.4.2 *Let $E = \mathbb{R}^3$, $\{e_i\}$ the canonical basis, and $x = x_1e_1 + x_2e_2 + x_3e_3$. Let:*

$$q(x) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 4x_2x_3$$

Choose a square term, e.g., x_1^2. Group terms with x_1:

$$q(x) = x_1^2 + 2x_1x_2 + 2x_2^2 + 5x_3^2 - 4x_2x_3$$

Complete the square:

$$= (x_1 + x_2)^2 - x_2^2 + 2x_2^2 + 5x_3^2 - 4x_2x_3 = (x_1 + x_2)^2 + x_2^2 + 5x_3^2 - 4x_2x_3$$

Now group terms with x_2:

$$= (x_1 + x_2)^2 + (x_2 - 2x_3)^2 - 4x_3^2 + 5x_3^2 = (x_1 + x_2)^2 + (x_2 - 2x_3)^2 + x_3^2$$

3.4.3 Classification of Quadratic Forms

Definition 3.4.2 *Let q be a quadratic form on a real vector space E. We say that:*

- *q is **positive semi-definite** (resp. negative semi-definite) if $q(x) \geq 0$ (resp. $q(x) \leq 0$) for all $x \in E$.*
- *q is **positive definite** (resp. negative definite) if $q(x) > 0$ (resp. $q(x) < 0$) for all $x \in E \setminus \{0\}$.*

Proposition 3.4.1 *If q is a definite quadratic form, then it is either positive definite or negative definite.*

Theorem 3.4.3 (Sylvester's Law of Inertia) *Let E be a real vector space of dimension n, and let q be a quadratic form on E. Then there exists a basis $\{e_1, \dots, e_n\}$ of E such that for $x = \sum_{i=1}^n x_ie_i$, we have:*

$$q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2$$

*where $r =$ rank(q) and $p \geq 0$ depends only on the form q, not on the basis. The pair $(p, r - p)$ is called the **signature** of q, denoted sign(q)*

The signature of a real quadratic form can be interpreted as follows: Let p be the largest dimension of a subspace of E on which q is positive definite, and $r-p$ the largest dimension of a subspace of E on which q is negative definite.

Corollaire 3.4.1 *Let q be a quadratic form on a real vector space E of finite dimension n. Then:*

- *q is positive definite $\iff$ sign$(q) = (n, 0)$*
- *q is negative definite $\iff$ sign$(q) = (0, n)$*
- *q is non-degenerate $\iff$ sign$(q) = (p, n-p)$*

Example 3.4.3

1. *Determine the signature of the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defined in the canonical basis by:*

$$q(x) = x_1^2 + 2x_2^2 + 15x_3^2 - 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

Applying the Gauss method, we find:

$$q(x) = (x_1 - 2x_2 + 3x_3)^2 - 2(x_2 - x_3)^2 + 8x_3^2$$

Let:

$$x_1' = x_1 - 2x_2 + 3x_3, \quad x_2' = \sqrt{2}(x_2 - x_3), \quad x_3' = \sqrt{8}x_3$$

Then:

$$q(x) = x_1'^2 - x_2'^2 + x_3'^2$$

So the signature is sign$(q) = (2, 1)$. According to Corollary 3.4.1, q is non-degenerate.

2. *In Example 3.4.1, we found sign$(q) = (3, 0)$. Therefore, q is positive definite.*

3.5 Exercises

In this section, we present a series of application-oriented exercises designed to illustrate the practical relevance of the main theoretical results established earlier.

3.5.1 Exercises Statements

Exercise 3.5.1 *Let φ be the symmetric bilinear form given by:*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

1. *Make φ explicit and determine the associated quadratic form q.*
2. *Determine its rank and kernel. Is it a non-degenerate form?*
3. *Find the orthogonal complement of $F = Span\{(1,0,0),(1,0,1)\}$.*

Exercise 3.5.2 *Let q be the quadratic form on $\mathbb{R}^4$ defined for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by:*

$$q(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

1. *Why is q a quadratic form?*
2. *What is its polar form? What is its matrix in the canonical basis of $\mathbb{R}^4$?*
3. *Decompose q into squares of linear forms and give its signature and rank.*

Exercise 3.5.3 *Let q be the quadratic form on $\mathbb{R}^4$ defined for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by:*

$$q(x) = 9x_1^2 + 5x_2^2 + 8x_2x_3 - 4x_2x_4 + 5x_3^2 + 4x_3x_4 + 8x_4^2$$

1. *Determine the kernel of q.*
2. *Provide a decomposition of q into squares; deduce its rank and signature. Is the form q degenerate?*
3. *Determine the isotropic vectors of q.*

Exercise 3.5.4 *Let q_α be the quadratic form defined on $\mathbb{R}^3$ by:*

$$q_\alpha(x) = x_1^2 + (1+\alpha)x_2^2 + (1+\alpha+\alpha^2)x_3^2 + 2x_1x_2 - 2\alpha x_2x_3, \quad \forall x \in \mathbb{R}^3$$

1. *Give the matrix M of q_α in the canonical basis of $\mathbb{R}^3$ and compute its determinant.*

2. *For which values of α is the form non-degenerate?*
3. *Give the signature and rank of q_α as a function of α.*

Exercise 3.5.5 *Let $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form defined by:*

$$\varphi(x,y) = \alpha x_1y_1 - x_1y_2 + 2x_1y_3 - x_2y_1 - 3x_2y_2 + 2x_2y_3 + 2x_3y_1 + 2x_3y_2 - x_3y_3, \qquad \alpha \in \mathbb{R}.$$

1. *Give the matrix of φ in the canonical basis of $\mathbb{R}^3$.*
2. *Determine the rank φ depending on the values of α. For which value(s) of α, φ is non-degenerate?*
3. *Give the associated quadratic form q using Gauss decomposition.*

3.5.2 Exercises Corrections

Solution 3.5.1

1. *Let $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then:*

$$\varphi(x,y) = x^tMy = \sum_{i,j=1}^{3} m_{ij}x_iy_j$$

Explicitly:

$$\varphi(x,y) = x_1y_1 + x_1y_2 + x_1y_3 + x_2y_1 + 2x_2y_2 + 3x_2y_3 + x_3y_1 + 3x_3y_2 + 5x_3y_3$$

The associated quadratic form is:

$$q(x) = \varphi(x,x) = x^tMx$$

Compute:

$$\begin{aligned} q(x) &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 6x_2x_3 + 5x_3^2 \\ &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 6x_2x_3 + 5x_3^2 \end{aligned}$$

2. *To determine the rank and kernel, we analyze the matrix M.*

   Rank:

   We perform row operations or compute the determinant of minors. Observed

$$\det(M) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1\cdot(2\cdot5-3\cdot3)-1\cdot(1\cdot5-3\cdot1)+1\cdot(1\cdot3-2\cdot1) = 1(10-9)-1(5-3)+1(3-2) = 1-2+$$

So the determinant is zero $\Rightarrow$ *matrix is not invertible* $\Rightarrow$ *rank* < 3

Try computing a 2×2 *minor:*

$$\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot 3 = 10 - 9 = 1 \neq 0$$

So rank is at least 2. Since the full determinant is zero, rank is exactly 2.

Thus $\operatorname{rank}(\varphi) = 2$.

Kernel:

The kernel of φ *is:*

$$\ker(\varphi) = \{x \in \mathbb{R}^3 \mid \varphi(x, y) = 0 \textit{ for all } y \in \mathbb{R}^3\} \Rightarrow Mx = 0$$

Solve $Mx = 0$*. Let* $x = (x_1, x_2, x_3)$*, then:*

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ x_1 + 3x_2 + 5x_3 = 0 \end{cases}$$

Subtract first from second:

$$(x_1 + 2x_2 + 3x_3) - (x_1 + x_2 + x_3) = x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$$

Subtract second from third:

$$(x_1 + 3x_2 + 5x_3) - (x_1 + 2x_2 + 3x_3) = x_2 + 2x_3 = 0 \Rightarrow same\ \ equation$$

Substitute into first:

$$x_1 + x_2 + x_3 = x_1 - 2x_3 + x_3 = x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

So:

$$x = (x_3, -2x_3, x_3) = x_3(1, -2, 1) \Rightarrow \ker(\varphi) = \textit{span}(1, -2, 1)$$

Thus, Kernel is 1-dimensional. Since kernel $\neq \{0\}$*,* φ *is degenerate.*

3. *Let* $F = \textit{Span}(v_1, v_2)$*, where:*

$$v_1 = (1, 0, 0), \quad v_2 = (1, 0, 1)$$

We seek:

$$F^{\perp} = \{x \in \mathbb{R}^3 \mid \varphi(x, v_1) = 0,\ \varphi(x, v_2) = 0\}$$

Let $x = (x_1, x_2, x_3)$. *Compute:*

$$\varphi(x, v_1) = x^t M v_1 = x_1 \cdot 1 + x_2 \cdot 1 + x_3 \cdot 1 = x_1 + x_2 + x_3$$

$$\varphi(x, v_2) = x^t M v_2 = x_1 \cdot 2 + x_2 \cdot 4 + x_3 \cdot 6 = 2x_1 + 4x_2 + 6x_3$$

So:

$$F^{\perp} = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0,\ 2x_1 + 4x_2 + 6x_3 = 0\}$$

Solve the system:

From first: $x_1 = -x_2 - x_3$ *Substitute into second:*

$$\begin{aligned} 2(-x_2 - x_3) + 4x_2 + 6x_3 &= -2x_2 - 2x_3 + 4x_2 + 6x_3 \\ &= 2x_2 + 4x_3 = 0 \\ &\Rightarrow x_2 = -2x_3 \\ &\Rightarrow x_1 = -(-2x_3) - x_3 = 2x_3 - x_3 = x_3 \end{aligned}$$

So:

$$x = (x_3, -2x_3, x_3) = x_3(1, -2, 1) \Rightarrow F^{\perp} = \mathit{span}(1, -2, 1)$$

Thus, the orthogonal complement of F is the line generated by $(1, -2, 1)$, *which coincides with the kernel of* φ.

Solution 3.5.2

1. *The function* q *is a homogeneous polynomial of degree 2 in the variables* x_1, x_2, x_3, x_4. *Therefore, it defines a quadratic form on* $\mathbb{R}^4$.

2. *The polar form* b *associated with* q *is defined by:*

$$\forall x, y \in \mathbb{R}^4, \quad b(x, y) = \frac{q(x + y) - q(x) - q(y)}{2}.$$

For all $x, y \in \mathbb{R}^4$, *we compute:*

$$b(x, y) = \frac{1}{2}\left[x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2 + x_2y_4 + x_4y_2 + x_3y_4 + x_4y_3\right].$$

The associated matrix M of the bilinear form b is:

$$M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

3. *The quadratic form q contains only cross-product terms (no squares). For example:*

$$q(x) = x_1x_2 + x_1(x_3 + x_4) + x_2(x_3 + x_4) + x_3x_4.$$

We rewrite:

$$q(x) = (x_1 + x_3 + x_4)(x_2 + x_3 + x_4) - (x_3 + x_4)^2 + x_3x_4.$$

Using the identity $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$, we obtain:

$$\begin{aligned} q(x) &= \frac{1}{4}(x_1 + x_2 + 2x_3 + 2x_4)^2 - \frac{1}{4}(x_1 - x_2)^2 - x_3^2 - x_3x_4 - x_4^2 \\ &= \frac{1}{4}(x_1 + x_2 + 2x_3 + 2x_4)^2 - \frac{1}{4}(x_1 - x_2)^2 - \left(x_3 + \frac{1}{2}x_4\right)^2 - \frac{3}{4}x_4^2. \end{aligned}$$

Therefore, the quadratic form q has rank 4, and is thus non-degenerate. Its signature is $(1, 3)$, meaning one positive and three negative eigenvalues.

Solution 3.5.3 1. ***Kernel of*** q

The kernel of q is defined by:

$$\ker q = \left\{ x \in \mathbb{R}^4 \mid b(x, y) = 0,\ \forall y \in \mathbb{R}^4 \right\},$$

where b is the polar form of q. Let

$$M = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 4 & -2 \\ 0 & 4 & 5 & 2 \\ 0 & -2 & 2 & 8 \end{pmatrix}$$

be the matrix of q in the canonical basis of $\mathbb{R}^4$. Then:

$$\ker q = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid Mx = 0 \right\}.$$

Solving the system $Mx = 0$:

$$\begin{cases} 9x_1 = 0 \\ 5x_2 + 4x_3 - 2x_4 = 0 \\ 4x_2 + 5x_3 + 2x_4 = 0 \\ -2x_2 + 2x_3 + 8x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_3 = -x_2 \\ x_3 = -2x_4 \end{cases} \Rightarrow x_2 = 2x_4, \quad x_3 = -2x_4, \quad x_1 = 0.$$

Thus:

$$\ker q = \{x = x_4(0,\ 2,\ -2,\ 1) \mid x_4 \in \mathbb{R}\}.$$

2. ***Decomposition into Squares of** q*

In the expression of $q(x)$, the variable x_1 appears only in the term $9x_1^2$, so we can write:

$$q(x) = 9x_1^2 + q_1(x_2, x_3, x_4),$$

where q_1 is a quadratic form in x_2, x_3, x_4. We compute:

$$\begin{aligned} q_1(x_2, x_3, x_4) &= 5x_2^2 + 4(2x_3 - x_4)x_2 + 5x_3^2 + 4x_3x_4 + 8x_4^2 \\ &= 5\left(x_2 + \frac{2}{5}(2x_3 - x_4)\right)^2 - \frac{4}{5}(2x_3 - x_4)^2 + 5x_3^2 + 4x_3x_4 + 8x_4^2. \end{aligned}$$

Simplifying:

$$q_1(x_2, x_3, x_4) = 5\left(x_2 + \frac{4}{5}x_3 - \frac{2}{5}x_4\right)^2 + \frac{9}{5}x_3^2 + \frac{36}{5}x_3x_4 + \frac{36}{5}x_4^2.$$

We recognize:

$$\frac{9}{5}x_3^2 + \frac{36}{5}x_3x_4 + \frac{36}{5}x_4^2 = \frac{9}{5}(x_3 + 2x_4)^2.$$

Therefore, for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we have:

$$q(x) = 9x_1^2 + 5\left(x_2 + \frac{4}{5}x_3 - \frac{2}{5}x_4\right)^2 + \frac{9}{5}(x_3 + 2x_4)^2.$$

This shows that q is a sum of squares of three independent linear forms. Hence:

$$\operatorname{rank}(q) = 3, \quad \operatorname{sign}(q) = (3, 0).$$

Since all coefficients are positive, q is positive definite but degenerate (not full rank).

3. ***Isotropic Cone of*** q

The isotropic cone of q *is defined by:*

$$C(q) = \left\{ x \in \mathbb{R}^4 \mid q(x) = 0 \right\}.$$

Since q *is positive definite, the only way for* $q(x) = 0$ *is for each squared term to vanish. From the decomposition:*

$$\begin{cases} x_1 = 0 \\ x_2 + \frac{4}{5}x_3 - \frac{2}{5}x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \quad \Rightarrow x_1 = 0, \quad x_3 = -2x_4, \quad x_2 = 2x_4.$$

Thus:

$$C(q) = \{ x = x_4(0,\ 2,\ -2,\ 1) \mid x_4 \in \mathbb{R} \}.$$

We observe that $\ker q = C(q)$*, since* q *is positive definite.*

Solution 3.5.4 1. ***Matrix Representation***

The matrix of the quadratic form q_α *in the canonical basis is:*

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1+\alpha & -\alpha \\ 0 & -\alpha & 1+\alpha+\alpha^2 \end{pmatrix}.$$

2. ***Determinant and Non-Degeneracy***

We compute the determinant:

$$\det(q_\alpha) = \alpha(1+\alpha^2).$$

Therefore, q_α *is non-degenerate if and only if* $\operatorname{rank}(M) = 3$*, i.e.,* $\det(q_\alpha) \neq 0$*. This occurs if and only if* $\alpha \neq 0$*.*

3. ***Decomposition into Squares of Linear Forms***

We expand:

$$\begin{aligned} q_\alpha(x) &= x_1^2 + (1+\alpha)x_2^2 + (1+\alpha+\alpha^2)x_3^2 + 2x_1x_2 - 2\alpha x_2x_3 \\ &= (x_1+x_2)^2 - x_2^2 + (1+\alpha)x_2^2 - 2\alpha x_2x_3 + (1+\alpha+\alpha^2)x_3^2 \\ &= (x_1+x_2)^2 + \alpha(x_2-x_3)^2 - \alpha x_3^2 + (1+\alpha+\alpha^2)x_3^2 \\ &= (x_1+x_2)^2 + \alpha(x_2-x_3)^2 + (1+\alpha^2)x_3^2. \end{aligned}$$

4. ***Rank and Signature Analysis***

   We distinguish three cases:

   - *If* $\alpha < 0$*, then the form has signature* $(2, 1)$ *and rank* 3*.*
   - *If* $\alpha = 0$*, then the form becomes:*

     $$q_0(x) = (x_1 + x_2)^2 + x_3^2,$$

     which has signature $(2, 0)$ *and rank* 2 *(hence degenerate).*
   - *If* $\alpha > 0$*, then all coefficients are positive, and the form has signature* $(3, 0)$ *and rank* 3*.*

Solution 3.5.5 *1. We express* $\varphi(x, y) = x^\top My$*, where* $M \in \mathbb{R}^{3\times3}$ *is the matrix of the bilinear form in the canonical basis.*

From the expression, we extract coefficients:

$$M = \begin{bmatrix} \alpha & -1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

2. *To determine the rank of* φ*, we compute the determinant of* M*:*

$$\det(M) = \begin{vmatrix} \alpha & -1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

We expand along the first row:

$$\begin{aligned} \det(M) &= \alpha \cdot \begin{vmatrix} -3 & 2 \\ 2 & -1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & -3 \\ 2 & 2 \end{vmatrix} \\ &= \alpha(-3 \cdot -1 - 2 \cdot 2) + 1(-1 \cdot -1 - 2 \cdot 2) + 2(-1 \cdot 2 - (-3) \cdot 2) \\ &= \alpha(3 - 4) + (1 - 4) + 2(-2 + 6) \\ &= \alpha(-1) - 3 + 8 = -\alpha + 5 \end{aligned}$$

Thus,

- $\mathrm{rank}(\varphi) = 3$ *if* $\det(M) \neq 0 \Rightarrow \alpha \neq 5$
- $\mathrm{rank}(\varphi) < 3$ *if* $\alpha = 5$

So:

- φ *is non-degenerate if and only if* $\alpha \neq 5$
- φ *is degenerate when* $\alpha = 5$

3. *We compute:*

$$q(x) = x^t M x = \sum_{i,j=1}^{3} m_{ij} x_i x_j$$

Explicitly:

$$q(x) = \alpha x_1^2 - 2x_1x_2 + 4x_1x_3 - 3x_2^2 + 4x_2x_3 - x_3^2$$

This is the symmetric quadratic form associated to φ*. To diagonalize it via Gauss (or completing the square), we proceed step-by-step:*

Let's write:

$$\begin{aligned} q(x) &= -[x_3^2 - 2(2x_1 + 2x_2)x_3] + \alpha x_1^2 - 3x_2^2 - 2x_1x_2 \\ &= -(2x_1 + 2x_2 - x_3)^2 + (\alpha + 4)x_1^2 + x_2^2 + 6x_1x_2 \\ &= (\alpha + 5)x_1^2 + (3x_1 + x_2)^2 - (2x_1 + 2x_2 - x_3)^2. \end{aligned}$$

Conclusion

In this chapter, we learned how quadratic forms are closely related to symmetric bilinear forms. We explored important concepts like rank, kernel, and isotropic vectors, and saw how to simplify quadratic forms using the Gauss method.

We also discovered how to classify quadratic forms based on their signature, which tells us whether a form is positive, negative, or indefinite. Sylvester's theorem helped us understand that every real quadratic form can be written as a sum of squares in a suitable basis.

These tools are very useful for solving problems in geometry, optimization, and many other areas of mathematics.

Chapter 4

Introduction to hermitian space

Introduction

In this chapter, we work over $\mathbb{C}$, the field of complex numbers. Our goal is to develop a theory of bilinear and quadratic forms that generalizes the classical real case. Within this framework, the complex conjugate $\overline{a}$ of a number $a \in \mathbb{C}$ plays a central role, replacing the notion of positivity that is naturally satisfied in the real setting.

Indeed, while the square x^2 of a real number is always non-negative, this property does not hold in $\mathbb{C}$. Instead, we rely on the modulus squared $|a|^2 = a\overline{a}$, which is always a real and non-negative quantity. This substitution allows us to define a coherent and meaningful theory of Hermitian forms, which serve as the complex analogues of real quadratic forms see [6].

4.1 Hermitian Form

Definition 4.1.1 (Semi-linear Maps) *Let E be a vector space over $\mathbb{C}$. A map $h : E \to E$ is said to be **semi-linear** if:*

(i) For all $x, y \in E$, $h(x+y) = h(x) + h(y)$

(ii) For all $x \in E$, $a \in \mathbb{C}$, $h(ax) = \overline{a}h(x)$

Definition 4.1.2 (Sesquilinear Forms) *A map $h : E \times E \to \mathbb{C}$ defined on a complex vector space E is called a **sesquilinear form** if:*

(i) $\forall x, y, z \in E$, $h(x, y + z) = h(x, y) + h(x, z)$

(ii) $\forall x, y, z \in E$, $h(x + z, y) = h(x, y) + h(z, y)$

(iii) $\forall x, y \in E, \forall a \in \mathbb{C}$, $h(ax, y) = \overline{a}h(x, y)$ *(semi-linearity in the first argument)*

(iv) $\forall x, y \in E, \forall a \in \mathbb{C}$, $h(x, ay) = ah(x, y)$ *(linearity in the second argument)*

Remark 4.1.1 1. *The term* sesquilinear *comes from Greek and means "one and a half linear".*

2. *The form h is not bilinear due to its non-linearity in the first argument.*

3. *A sesquilinear form can be defined with the opposite convention: linear in the first argument and semi-linear in the second.*

Definition 4.1.3 (Hermitian Symmetric Sesquilinear Forms) *A sesquilinear form h is said to be* ***Hermitian symmetric*** *if:*

$$h(x, y) = \overline{h(y, x)} \quad \text{for all } x, y \in \mathbb{C}$$

That is, $h : E \times E \to \mathbb{C}$ is Hermitian symmetric if and only if:

(i) $\forall x, y, z \in E$, $h(x, y + z) = h(x, y) + h(x, z)$

(ii) $\forall x, y, z \in E$, $h(x + z, y) = h(x, y) + h(z, y)$

(iii) $\forall x, y \in E, \forall a \in \mathbb{C}$, $h(ax, y) = \overline{a}h(x, y)$

(iv) $\forall x, y \in E, \forall a \in \mathbb{C}$, $h(x, ay) = ah(x, y)$

(v) $\forall x, y \in E$, $h(x, y) = \overline{h(y, x)}$

Remark 4.1.2 1. *Symmetric bilinear forms are special cases of Hermitian symmetric sesquilinear forms.*

2. *An example of a sesquilinear form on $\mathbb{C}^n$ is:*

$$h(x, y) = \sum_{i=1}^{n} \overline{x_i} y_i = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n, \qquad \forall x, y \in \mathbb{C}^n.$$

This is called the canonical sesquilinear form*, and it is Hermitian symmetric.*

Definition 4.1.4 (Hermitian Forms) *Let h be a Hermitian symmetric sesquilinear form on a complex vector space E. The associated* ***Hermitian form*** *is the map:*

$$q : E \to \mathbb{R}, \quad x \mapsto q(x) = h(x, x).$$

The form h is called the polar form *of q.*

Proprieties 4.1.1 *Let E be a complex vector space. A map $q : E \to \mathbb{R}$ is a Hermitian form if and only if:*

1. *For all $x \in E$, $\lambda \in \mathbb{C}$, we have:*

$$q(\lambda x) = |\lambda|^2 q(x)$$

2. *For all $x, y \in E$, we have:*

$$h(x, y) = \frac{1}{2}\left[q(x + y) - q(x) - q(y)\right] - \frac{i}{2}\left[q(x + iy) - q(x) - q(y)\right]$$

Just as in the real case, the Hermitian form q and its polar form h are equivalent, and studying one amounts to studying the other.

4.2 Matrix Representation

Definition 4.2.1 (Hermitian Matrices) *Let $H \in \mathcal{M}_n(\mathbb{C})$. The matrix H is called* ***Hermitian*** *if:*

$$\overline{H}^T = H$$

In particular, real symmetric matrices are real Hermitian matrices.

Remark 4.2.1 *The diagonal elements of a Hermitian matrix are real.*

Example 4.2.1 *The complex matrix H defined below is Hermitian:*

$$H = \begin{pmatrix} 1 & -i & 3+2i \\ i & 5 & 1+i \\ 3-2i & 1-i & 4 \end{pmatrix}$$

Just as symmetric bilinear forms are represented by real symmetric matrices, Hermitian forms are represented by Hermitian matrices.

Let E be a complex vector space with basis $\{e_i\}$, and let $x, y \in E$ such that:

$$x = \sum_{i=1}^{n} x_i e_i, \quad y = \sum_{j=1}^{n} y_j e_j$$

Then:

$$h(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{x_i} y_j h(e_i, e_j)$$

Definition 4.2.2 (Matrix of a Hermitian Form) *Let h be a Hermitian form defined on a finite-dimensional complex vector space E, and let $\{e_i\}$ be a basis of E. The* ***matrix of*** *h in the basis $\{e_i\}$ is the Hermitian matrix:*

$$M_h = \begin{pmatrix} h(e_1, e_1) & \cdots & h(e_1, e_n) \\ \vdots & \ddots & \vdots \\ h(e_n, e_1) & \cdots & h(e_n, e_n) \end{pmatrix}$$

The element $h(e_i, e_j)$ is the coefficient of $\overline{x_i} y_j$ in the expression of the Hermitian form h.

Example 4.2.2

1. *The square of the modulus of a linear form on a complex vector space E is a Hermitian form. Let $\varphi \in E^*$. The map:*

   $$q(x) = |\varphi(x)|^2 = \overline{\varphi(x)}\varphi(x)$$

   is a Hermitian form whose polar form is:

   $$h(x, y) = \overline{\varphi(x)}\varphi(y)$$

2. *The map $h : \mathbb{C}^2 \to \mathbb{C}$ defined by:*

   $$h(x, y) = \overline{x_1} y_1 + 5\overline{x_2} y_2 + (3 + i)\overline{x_1} y_2 + (3 - i)\overline{x_2} y_1$$

   is a Hermitian form. Its associated Hermitian form is:

   $$q(x) = |x_1|^2 + 5|x_2|^2 + (3 + i)\overline{x_1} x_2 + (3 - i)\overline{x_2} x_1$$

Remark 4.2.2 *Let E be a complex vector space and h a Hermitian form on E. One can define, analogously to real quadratic forms, the notions of conjugate vectors, orthogonality, kernel, and isotropic vectors for Hermitian forms.*

4.3 Hermitian Space

Definition 4.3.1 (Hermitian Inner Product) *A **Hermitian inner product** on a complex vector space E, denoted $\langle \cdot, \cdot \rangle$, is a positive-definite Hermitian form. That is, it satisfies the following properties:*

(i) $\forall x, y, z \in E$, $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(ii) $\forall x, y, z \in E$, $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(iii) $\forall x, y \in E, \forall a \in \mathbb{C}$, $\langle ax, y \rangle = \overline{a} \langle x, y \rangle$

(iv) $\forall x, y \in E, \forall a \in \mathbb{C}$, $\langle x, ay \rangle = a \langle x, y \rangle$

(v) $\forall x, y \in E$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(vi) $\forall x \in E$, $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0 \Rightarrow x = 0$

Definition 4.3.2 (Hermitian Space) *A **Hermitian space** is a finite-dimensional complex vector space E equipped with a Hermitian inner product.*

Remark 4.3.1 *A Hermitian space is also called a* Hilbert space. *If the complex vector space E with a Hermitian inner product is infinite-dimensional, it is called a* pre-Hilbert space.

Definition 4.3.3 (Hermitian Norm) *Let E be a complex vector space with a Hermitian inner product. The **Hermitian norm** is the map $\|\cdot\| : E \to \mathbb{R}^+$ defined by:*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

It satisfies:

(i) $\|x\| = 0 \Rightarrow x = 0$

(ii) $\|ax\| = |a| \cdot \|x\|$ *for all* $a \in \mathbb{C}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

Proposition 4.3.1 (Cauchy-Schwarz Inequality) *Let E be a Hermitian space. For all $x, y \in E$, we have:*

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Equality holds if and only if x and y are linearly dependent.

4.4 Exercises on Hermitian Forms

In this section, we present a series of application-oriented exercises designed to illustrate the practical relevance of the main theoretical results established earlier.

Exercise 4.4.1 (Sesquilinearity Check) *Let* $h : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ *be defined by:*

$$h(x, y) = \overline{x_1} y_1 + 2\overline{x_2} y_2 + \overline{x_1} y_2$$

Verify whether h *is a sesquilinear form.*

Solution 4.4.1 *We check the four properties:*

- *Additivity in each argument: follows from linearity of addition.*
- *Semi-linearity in the first argument:*

$$h(ax, y) = \overline{ax_1} y_1 + 2\overline{ax_2} y_2 + \overline{ax_1} y_2 = \overline{a} h(x, y)$$

- *Linearity in the second argument:*

$$h(x, ay) = \overline{x_1} a y_1 + 2\overline{x_2} a y_2 + \overline{x_1} a y_2 = a h(x, y)$$

Therefore, h *is sesquilinear.*

Exercise 4.4.2 (Hermitian Symmetry) *Let* $h(x, y) = \overline{x_1} y_1 + \overline{x_2} y_2 + i\overline{x_1} y_2 - i\overline{x_2} y_1$*. Is* h *Hermitian symmetric?*

Solution 4.4.2 *We compute:*

$$h(y, x) = \overline{y_1} x_1 + \overline{y_2} x_2 + i\overline{y_1} x_2 - i\overline{y_2} x_1$$

Taking the complex conjugate:

$$\overline{h(y, x)} = \overline{\overline{y_1} x_1 + \overline{y_2} x_2 + i\overline{y_1} x_2 - i\overline{y_2} x_1} = \overline{x_1} y_1 + \overline{x_2} y_2 + i\overline{x_1} y_2 - i\overline{x_2} y_1 = h(x, y)$$

Thus, h *is Hermitian symmetric.*

Exercise 4.4.3 (Matrix Representation) *Let $E = \mathbb{C}^2$ with basis $\{e_1, e_2\}$, and let h be defined by:*

$$h(x, y) = \overline{x_1}y_1 + 3\overline{x_2}y_2 + 2\overline{x_1}y_2 + 2\overline{x_2}y_1$$

Find the matrix M_h of h in the basis $\{e_1, e_2\}$.

Solution 4.4.3 *We compute:*

$$h(e_1, e_1) = 1, \quad h(e_1, e_2) = 2, \quad h(e_2, e_1) = 2, \quad h(e_2, e_2) = 3$$

So:

$$M_h = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

This matrix is Hermitian.

Exercise 4.4.4 (Hermitian Norm) *Let $\langle x, y\rangle = \overline{x_1}y_1 + \overline{x_2}y_2$. Compute $\|x\|$ for $x = (3+i, 1-i)$.*

Solution 4.4.4

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{(3 - i)(3 + i) + (1 + i)(1 - i)} = \sqrt{10}$$

Exercise 4.4.5 (Cauchy-Schwarz Inequality) *Let $x = (1, i)$, $y = (2, 1)$, and $\langle x, y\rangle = \overline{x_1}y_1 + \overline{x_2}y_2$. Verify the Cauchy-Schwarz inequality.*

Solution 4.4.5

$$\langle x, y\rangle = \overline{1} \cdot 2 + \overline{i} \cdot 1 = 2 - i \Rightarrow |\langle x, y\rangle| = \sqrt{5}$$

$$\|x\| = \sqrt{1 + 1} = \sqrt{2}, \quad \|y\| = \sqrt{4 + 1} = \sqrt{5} \Rightarrow \|x\| \cdot \|y\| = \sqrt{10}$$

Since $\sqrt{5} \leq \sqrt{10}$, the inequality holds.

Conclusion

The equivalence between Hermitian forms and their polar forms, along with their matrix representations, provides a concrete computational approach. Moreover, the introduction of Hermitian spaces and the Cauchy-Schwarz inequality lays the groundwork for deeper studies in functional analysis, quantum mechanics, and complex geometry.

This chapter establishes the essential vocabulary and structure for working with complex inner product spaces, preparing the reader for advanced topics such as spectral theory, unitary transformations, and Hilbert space analysis.

Chapter 5

Spectral decomposition of self-adjoint linear mappings

Introduction

Spectral decomposition is a fundamental concept in linear algebra and functional analysis, particularly for understanding the structure of self-adjoint (Hermitian) operators on complex inner product spaces. It allows such operators to be expressed in terms of their eigenvalues and eigenvectors, facilitating both theoretical analysis and practical computation see [7].

5.1 Self-Adjoint Operators

At first, define unitary and normal matrices or linear mapping.

Definition 5.1.1 *Let* $A \in \mathcal{M}_n(\mathbb{C})$.

1. A *is said to be* ***unitary*** *if* $A^* = A^{-1}$.
2. A *is said to be* ***normal*** *if* $A^*A = AA^*$. *This means that* A *commutes with its conjugate transpose.*

Example 5.1.1 *Consider the matrices*

$$U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -i & -i \\ -i & i \end{pmatrix}.$$

We can verify that U is unitary, since $U^* = U^{-1}$. On the other hand, we can easily check that $N^*N = NN^*$, so N is normal.

Proposition 5.1.1 *Every $n \times n$ complex invertible matrix A can be represented as*

$$A = U \cdot T,$$

where U is unitary and $T = (t_{ij})$ is upper triangular with $t_{ij} \geq 0$.

Proof. The proof is similar to the real case. ■

Lemma 5.1.1 *Every Hermitian matrix $A \in \mathcal{M}_n(\mathbb{R})$ can be represented in the form:*

$$A = P^t D \overline{P},$$

where P is orthogonal and D is diagonal, with diagonal entries in $\mathbb{R}$ being the eigenvalues of A.

From the above lemma, we deduce that every Hermitian positive definite matrix A can be written as:

$$A = M^t \overline{M},$$

where $M = \sqrt{D}P$ is invertible.

Definition 5.1.2 *Let $f \in \mathcal{L}(E)$. The* ***adjoint*** *(or Hermitian conjugate) of f is the mapping $f^* \in E^*$ satisfying:*

$$\langle f(u), v\rangle = \langle u, f^*(v)\rangle,$$

for any $u, v \in E$. Furthermore, f is said to be ***self-adjoint*** *or* ***Hermitian*** *if $f = f^*$.*

Theorem 5.1.1 *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix (or equivalently, a self-adjoint mapping). Then $x^t A\overline{x} \in \mathbb{R}$ for each $x \in \mathbb{C}^n$.*

Proof. We have:

$$\begin{aligned}
x^t A\overline{x} &= (x^t A\overline{x})^t \quad \text{(since } x^t A\overline{x} = a \in \mathbb{C}\text{)},\\
&= (\overline{x})^t A^t x\\
&= (\overline{x})^t \overline{A^*} x\\
&= \overline{x^t A^* \overline{x}}\\
&= \overline{x^t A\overline{x}} \quad \text{(since } A^* = A\text{)}
\end{aligned}$$

This implies that $x^tA\overline{x} = \overline{x^tA\overline{x}}$, hence $x^tA\overline{x} \in \mathbb{R}$.

Second Proof: We know that:

$$x^tA\overline{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{pmatrix},$$

where $a_{ii} \in \mathbb{R}$ for $1 \le i \le n$, and $a_{ij} = \overline{a_{ji}}$ for $i \ne j$, because the matrix A is Hermitian. We expand the quadratic form $x^tA\overline{x}$ as follows:

$$\begin{aligned} x^tA\overline{x} &= \sum_{i,j} a_{ij}x_i\overline{x}_j \\ &= \sum_{i=1}^n a_{ii}x_i\overline{x}_i + \sum_{i\ne j} a_{ij}x_i\overline{x}_j \\ &= \underbrace{\sum_{i=1}^n a_{ii}|x_i|^2}_{\in\mathbb{R}} + \sum_{i<j} (a_{ij}x_i\overline{x}_j + a_{ji}x_j\overline{x}_i) \\ &= \underbrace{\sum_{i=1}^n a_{ii}|x_i|^2}_{\in\mathbb{R}} + \sum_{i<j} (a_{ij}x_i\overline{x}_j + \overline{a_{ij}x_j\overline{x}_i}) \\ &= \sum_{i=1}^n a_{ii}|x_i|^2 + 2\,\mathrm{Re}\left(\sum_{i<j} a_{ij}x_i\overline{x}_j\right) \in \mathbb{R}. \end{aligned}$$

■

Remark 5.1.1 *As a second method, we prove that the eigenvalues of a Hermitian matrix A (or self-adjoint mapping) are real. Let f_A be the Hermitian sesquilinear form associated with A, and let (λ, x) be an eigenpair of A. Applying Theorem5.1.1, we obtain:*

$$\underbrace{|f_A(x,x)|}_{\in\mathbb{R}} = x^tA\overline{x} = \overline{(x^t)}A\overline{\overline{x}} = \overline{x}^tAx = x^t(\lambda x) = \lambda\overline{x}^tx = \underbrace{\lambda\sum_{i=1}^n |x_i|^2}_{\in\mathbb{R}}.$$

Hence, $\lambda \in \mathbb{R}$.

Theorem 5.1.2 *The eigenvalues of a Hermitian matrix (or self-adjoint mapping) are real numbers.*

Proof. Let (λ, x) be an eigenpair of a Hermitian matrix A, with $x \neq 0$. Then:

$$\begin{aligned}\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Ax, x\rangle = (Ax)^t\overline{x} = x^t A^t \overline{x} \\ = x^t(\overline{A}^t)^t\overline{x} \quad \text{(since } \overline{A}^t = A\text{)} \\ = x^t\overline{A}\overline{x} = \langle x, Ax\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle.\end{aligned}$$

Thus, $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$. ■

5.2 Spectral Properties and Algebraic Comparison

Corollaire 5.2.1 *The eigenvalues of a real anti-symmetric matrix are purely imaginary.*

Proof. *First Method.*

It suffices to show that iA is Hermitian. Indeed,

$$(iA)^* = (\overline{(iA)})^t = -iA^t = -i(-A) = iA,$$

but since A is anti-symmetric, $A^t = -A$, so:

$$(iA)^* = iA.$$

Thus, iA is Hermitian. By Theorem 5.1.2, the eigenvalues of iA are real, and hence the eigenvalues of A are purely imaginary.

Second Method.

Proceeding as in the proof of Theorem 5.1.1 , let (λ, x) be an eigenpair of A. Then:

$$\begin{aligned}\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Ax, x\rangle = (Ax)^t\overline{x} = x^t A^t \overline{x} \\ = x^t(-\overline{A})\overline{x} = -x^t\overline{Ax} = \langle x, Ax\rangle = -\langle x, \lambda x\rangle = -\overline{\lambda}\langle x, x\rangle.\end{aligned}$$

Therefore,

$$(\lambda + \overline{\lambda})\langle x, x\rangle = 0.$$

Since $x \neq 0$, we deduce that $2\,\mathrm{Re}(\lambda) = 0$, hence λ is purely imaginary. ■

Theorem 5.2.1 (Spectral Decomposition of Self-Adjoint Mapping.) *Let E be a pre-Hilbert space over $\mathbb{C}$ with $\dim E = n$, and let $f \in \mathcal{L}(E)$. Then f is normal if and only if there exists an orthonormal basis of E formed by the eigenvectors of f.*

5.3 Exercises and Solution

Exercise 5.3.1 *Let*

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}.$$

1. *Show that A is self-adjoint.*
2. *Find its spectral decomposition.*

Solution 5.3.1 1. *A matrix is self-adjoint (symmetric) if $A = A^t$. Here,*

$$A^t = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} = A.$$

So A is self-adjoint.

2. *Compute the eigenvalues:*

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 1 = \lambda^2 - 7\lambda + 11.$$

Solve $\lambda^2 - 7\lambda + 11 = 0$. The roots are:

$$\lambda = \frac{7 \pm \sqrt{49-44}}{2} = \frac{7 \pm \sqrt{5}}{2}.$$

Let $\lambda_1 = \frac{7+\sqrt{5}}{2}$, $\lambda_2 = \frac{7-\sqrt{5}}{2}$. For each eigenvalue, find the corresponding eigenvector.

For λ_1, solve $(A - \lambda_1 I)v = 0$. The eigenvector is proportional to:

$$v_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1 - 4}{1} \end{pmatrix}.$$

Normalize v_1 to get an orthonormal basis.

Similarly for λ_2, get v_2. Then the spectral decomposition is:

$$A = \lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t,$$

where u_1, u_2 are the normalized eigenvectors.

Exercise 5.3.2 *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

Find an orthonormal basis of eigenvectors and express $A = PDP^t$*, where* D *is diagonal and* P *is orthogonal.*

Solution 5.3.2

- *A is symmetric, so it is diagonalizable with an orthonormal basis of eigenvectors.*
- *Compute the characteristic polynomial:*

$$\det(A - \lambda I) = (2 - \lambda) \det \begin{pmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{pmatrix} = (2 - \lambda)((5 - \lambda)^2 - 1).$$

- *Solve for eigenvalues:* $\lambda = 2, 4, 6$.
- *Find eigenvectors for each eigenvalue and orthonormalize them.*
- *Construct* $P = [u_1, u_2, u_3]$*, and* $D =$ *diag*$(2, 4, 6)$.
- *Then* $A = PDP^t$.

Exercise 5.3.3 *Let*

$$A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use the spectral theorem to find the decomposition of A *into orthogonal projections.*

Solution 5.3.3

- *A is symmetric, so it is self-adjoint.*
- *Compute eigenvalues: solve* $\det(A - \lambda I) = 0$.
- *Eigenvalues:* $\lambda_1 = 7, \lambda_2 = 2, \lambda_3 = 1$.
- *Find orthonormal eigenvectors* u_1, u_2, u_3.

- *Then the spectral decomposition is:*

$$A = \sum_{i=1}^{3} \lambda_i u_i u_i^t.$$

Exercise 5.3.4 *Let*

$$Q(x) = x^t A x, \quad A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

Diagonalize A and express $Q(x)$ as a sum of squares.

Solution 5.3.4

- *A is symmetric $\Rightarrow$ diagonalizable.*
- *Compute eigenvalues:*

$$\det(A - \lambda I) = (3 - \lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 2, 4.$$

- *Find orthonormal eigenvectors u_1, u_2.*
- *Let $P = [u_1,\ u_2]$, then $A = PDP^t$.*
- *Let $y = P^t x$, then:*

$$Q(x) = x^t A x = x^t P D P^t x = y^t D y = \lambda_1 y_1^2 + \lambda_2 y_2^2.$$

- *Thus, $Q(x)$ is a sum of squares in the new orthonormal basis.*

Conclusion

Spectral decomposition provides a profound insight into the structure of self-adjoint linear operators. By expressing such operators as weighted sums of orthogonal projections onto their eigenspaces, we gain both theoretical clarity and computational efficiency. The spectral theorem guarantees the existence of a complete orthonormal basis of eigenvectors and ensures that all eigenvalues are real, making self-adjoint operators particularly well-behaved.

This decomposition is not only central to pure mathematics but also foundational in applied fields such as quantum mechanics, signal processing, and numerical analysis. Through spectral decomposition, complex transformations become transparent, and the behavior of linear systems can be understood in terms of their simplest components.

CONCLUSION

This course has provided a rigorous and structured exploration of advanced linear algebra concepts, focusing on bilinear and quadratic forms, sesquilinear structures, and the foundations of Hermitian geometry. Beginning with the duality of vector spaces and linear forms, we developed a deep understanding of how algebraic operations reflect geometric intuition.

Through the study of bilinear forms, we examined how symmetry, alternation, and non-degeneracy shape the structure of vector spaces, leading to powerful tools like the Gauss decomposition and the construction of orthogonal bases. The transition to quadratic forms allowed us to classify and decompose expressions into canonical forms, revealing their signatures and applications in number theory, geometry, and physics.

The introduction of sesquilinear and Hermitian forms extended these ideas into complex vector spaces, laying the groundwork for Hermitian geometry, which plays a central role in modern mathematics and quantum theory. These forms preserve inner product structures and enable the spectral analysis of self-adjoint operators, culminating in the spectral decomposition theorem.

Altogether, this course has not only equipped students with essential computational techniques but also fostered a conceptual framework that unifies algebraic and geometric reasoning. It opens the door to further studies in functional analysis, differential geometry, and mathematical physics, where these foundational ideas continue to resonate.

Bibliography

[1] I.Kaplansky, Linear Algebra and Geometry, A Second Course," Allyn and Bacon, Boston,1969.

[2] Kurgalin, Sergei, et al., Bilinear and Quadratic Forms. Algebra and Geometry with Python (2021), 335-356.

[3] M.Knebusch, Specialization of Quadratic and Symmetric Bilinear Forms (Vol. 11). Springer Science and Business Media, 2011.

[4] Milnor, J. W., and Husemoller, D. (1973). Symmetric bilinear forms (Vol. 5). Berlin:Springer.

[5] A.M. Robert, Linear Algebra : Examples and Applications. World Scientific, 2005.

[6] W.Scharlau, Quadratic and Hermitian Forms, Springer-Verlag, Berlin, 1985.

[7] D.Shapiro, Compositions of Quadratic Forms, de Gruyter, Berlin, 2000.

[8] K. Szymiczek, Ten problems on quadratic forms. Acta Mathematica et Informatica Universitatis Ostraviensis, 10(1)(2002), 133-143.

Contents