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Polynomial Decay of the Energy of Solutions of the Timoshenko System with Two Boundary Fractional Dissipations

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Abstract: In this study, we examine Timoshenko systems with boundary conditions featuring two types of fractional dissipations. By applying semigroup theory, we demonstrate the existence and uniqueness of solutions. Our analysis shows that while the system exhibits strong stability, it does not achieve uniform stability. Consequently, we derive a polynomial decay rate for the system.

Keywords: Timoshenko systems; fractional derivatives; $\mathcal{C}_0$ semi-group

MSC: 35L05; 34K35; 93D20

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1. Introduction

In this study, we investigate the well-posedness and stabilization of a one-dimensional Timoshenko system of the following form:

$$\begin{cases} \rho_1\varphi_{tt} - d_1(\varphi_x+\psi)_x = 0, & (x,t)\in(0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - d_2\psi_{xx} + d_1(\varphi_x+\psi) = 0, & (x,t)\in(0,L)\times(0,+\infty). \end{cases} \tag{1}$$

The initial conditions are

$$\begin{aligned} &\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad x\in(0,L), \\ &\psi(x,0)=\psi_0(x), \quad \psi_t(x,0)=\psi_1(x), \quad x\in(0,L), \end{aligned} \tag{2}$$

and the following boundary conditions:

$$\begin{aligned} &\varphi(0,t)=0, \quad \psi(0,t)=0, \quad \text{in } (0,+\infty), \\ &(\varphi_x+\psi)(L) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L), \quad (\psi_x+\psi)(L) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L), \quad \text{in } (0,+\infty), \end{aligned} \tag{3}$$

where $\rho_1, \rho_2, d_1, d_2, \gamma_1$, and γ_2 are positive constants, η is a non-negative constant, and α is in $(0,1)$.

The notation $\partial_t^{\alpha,\eta}$ represents the generalized Caputo fractional derivative of order α (where $0<\alpha<1$) with respect to time t. It is defined as

$$\partial_t^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)}\frac{df}{ds}(s)\,ds, \quad \eta\ge 0.$$

The Timoshenko system, traditionally used to model the behavior of beams in mechanical structures, is extended in this study to incorporate fractional derivatives on the boundary conditions. This extension is significant because fractional derivatives are known to provide more accurate models for systems with memory effects and complex dissipation properties, which are often encountered in practical applications, such as material microstructure analysis. The introduction of fractional derivatives into the boundary conditions is novel in the context of the Timoshenko system. This allows us to model more realistic dissipative effects that occur in various materials and structures. Our results provide new insights into the stability characteristics of systems with fractional boundary dissipation, contributing to both the theoretical understanding and practical applications in engineering and materials science.

Fractional calculus has developed into a well-established theory with a solid mathematical foundation, and its applications have gained significant interest in various research fields, including electrical circuits, chemical processes, signal processing, bioengineering, viscoelasticity, and control systems (see [1]). Fractional-order control is not only theoretically important but also has practical implications. It generalizes classical integer-order control theory, enabling more accurate modeling and enhanced control performance. Experimental observations reveal that many phenomena cannot be fully described using traditional Newtonian terms. For example, in viscoelastic materials, the material's microstructure leads to a combined response involving both elastic solid and viscous fluid characteristics.

The literature (see [2]) establishes that the fractional derivative ∂_t^α enforces dissipation in the system and ensures that the solution converges to an equilibrium state. Consequently, when applied at the boundaries, fractional derivatives can act as controllers to suppress or attenuate undesirable vibrations.

In [3], B. Mbodje explored the asymptotic behavior of solutions with the following system:

$$\begin{cases} \partial_t^2 u(x,t) - u_x^2(x,t) = 0, \quad (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = 0, \\ \partial_x u(1,t) = -k\partial_t^{\alpha,\eta} u(1,t), \quad \alpha \in (0,1), \eta \geq 0, k > 0, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = v_0(x). \end{cases}$$

He demonstrated strong asymptotic stability of the solutions when $\eta = 0$ and a polynomial decay rate of t^{-1} as time approaches infinity when $\eta \neq 0$. The polynomial decay rate was established using the energy method.

Kim and Renardy [4] investigated (1) with two boundary controls of the following form:

$$K(\varphi_x + \psi)(L) = -\gamma_1 \partial^\alpha t \varphi(L), \quad b\psi x(L) = -\gamma_2 \partial_t^\alpha \psi(L), \quad \text{for}, , t \in (0,+\infty),$$

and employed multiplier techniques to prove an exponential decay result for the natural energy of (1). Additionally, Yan [5] established a polynomial decay result when examining two boundary frictional damping terms with polynomial growth near the origin.

Benaissa and Benazzouz [6] investigated the stabilization of the following Timoshenko system with two dynamic boundary control conditions involving fractional derivatives:

$$\begin{cases} \rho_1 \phi_{tt} - K(\phi_x + \psi)x & = & 0, \quad (x,t) \in (0,L) \times (0,+\infty), \\ \rho 2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) & = & 0, \quad (x,t) \in (0,L) \times (0,+\infty). \end{cases}$$

The system is subject to the following boundary conditions:

$$\begin{cases} m_1 \phi_{tt}(L,t) + K(\phi_x + \psi)(L,t) & = & -\gamma_1 \partial_t^{\alpha,\eta} \phi(L,t) \text{ for } t \in (0,+\infty), \\ m_2 \psi_{tt}(L,t) + b\psi_x(L,t) & = & -\gamma_2 \partial_t^{\alpha,\eta} \psi(L,t) \text{ for } t \in (0,+\infty), \\ \phi(0,t) = 0, \quad \psi(0,t) & = & 0 \text{ for } t \in (0,+\infty), \end{cases}$$

where m_1 and m_2 are positive constants. They demonstrated that the system (1) is not uniformly stable using the spectrum method. Polynomial stability was established through semigroup theory and by applying a result from Borichev and Tomilov.

M. Akil et al. [7] studied the Timoshenko system with a single fractional derivative described by

$$\begin{cases} au_{tt} - (u_x + y)_x &= 0, \quad (x,t) \in (0,1) \times (0,+\infty), \\ by_{tt} - y_{xx} + c(u_x + y) &= 0, \quad (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{4}$$

where a, b, and c are positive constants. The system is subject to the following boundary conditions:

$$\begin{cases} u_x(1,t) + y(1,t) + \gamma \partial_t^{\alpha,\eta} u(1,t) &= 0, \quad t \in \mathbb{R}^+, \\ u(0,t) = y_x(0,t) = y_x(1,t) &= 0. \end{cases} \tag{5}$$

They demonstrated that the energy of the system (4) and (5) decays polynomially over time. References such as [8–14] present a comprehensive collection of published works that support the mathematical formulation of problems related to fractional differential equations and the decay rate of the associated energy.

This paper is organized as follows: in Section 2, we demonstrate the well-posedness of system (1) with the boundary conditions (3) using semigroup theory. In Section 3, we prove that the Timoshenko system (1) with the boundary conditions (2) is not exponentially stable, whether the wave propagation speeds are equal ($\frac{\rho_1}{d_1} = \frac{\rho_2}{d_2}$) or not ($\frac{\rho_1}{d_1} \neq \frac{\rho_2}{d_2}$). In Section 5, we show that the solution decays polynomially to zero when $\eta > 0$, employing a frequency domain approach and a theorem by Borichev and Tomilov.

2. Augmented Model and Well-Posedness of the System

In this section, we focus on reformulating the model (1) into an augmented system. To proceed, we first require the following theorem:

Theorem 1 ([3]). *Let μ be the following function:*

$$\mu(\xi) = |\xi|^{(2\alpha - n)/2}, \quad \xi \in \mathbb{R}^n, \quad 0 < \alpha < 1.$$

Consider the system governed by the equation:

$$\partial_t \varphi(\xi,t) + \left(|\xi|^2 + \eta\right)\varphi(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^n, \, \eta \geq 0, \, t > 0,$$

with the initial condition

$$\varphi(\xi, 0) = 0,$$

and the output defined as

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{n}{2}+1\right)}{n\pi^{\frac{n}{2}+1}} \int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,t)d\xi.$$

The relationship between the 'input' U and the 'output' O is then given by

$$O(t) = I^{1-\alpha,\eta}U(t) = D^{\alpha,\eta}U(t),$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau) d\tau.$$

Lemma 1 ([15]). *If $\lambda \in D = \{\lambda \in \mathbb{C} \mid \Re(\lambda) + \eta > 0\} \cup \{\Im(\lambda) \neq 0\}$, then*

$$\tau(\alpha) \int_{\mathbb{R}^n} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} d\xi = (\lambda + \eta)^{\alpha - 1},$$

where $\tau(\alpha) = \pi^{-1}\sin(\alpha\pi)$.

Using the previous theorem, the system (1) can be rewritten as the following augmented model:

$$\begin{cases} \rho_1\varphi_{tt} - d_1(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - d_2\psi_{xx} + d_1(\varphi_x+\psi) = 0, \quad (x,t) \in (0,L)\times(0,+\infty), \\ \partial_t\phi_1(\xi,t) + (\xi^2+\eta)\phi_1(\xi,t) - \mu(\xi)\partial_t\varphi(L,t) = 0, \quad t\in(0,+\infty), \quad \xi\in\mathbb{R}, \\ \partial_t\phi_2(\xi,t) + (\xi^2+\eta)\phi_2(\xi,t) - \mu(\xi)\partial_t\psi(L,t) = 0, \quad t\in(0,+\infty), \quad \xi\in\mathbb{R}, \\ (\varphi_x+\psi)(L) = -\gamma_1\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_{\mathbb{R}}\mu(\xi)\phi_1(\xi,t)d\xi, \quad t\in(0,+\infty), \\ \psi_x(L) = -\gamma_2\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_{\mathbb{R}}\mu(\xi)\phi_2(\xi,t)d\xi, \quad t\in(0,+\infty), \end{cases} \tag{6}$$

with the following initial conditions:

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x\in(0,L), \\ \psi(x,0) &= \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x\in(0,L). \end{aligned}$$

For a solution $U = (\varphi, \varphi_t, \psi, \psi_t, \phi_1, \phi_2)$ of (6), we define the energy by

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{7}$$

where

$$\begin{aligned} \|U\|^2_{\mathcal{H}} &= \frac{1}{2}\int_0^L \Big(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + d_1|\varphi_x+\psi|^2 + d_2|\psi_x|^2\Big)dx \\ &+ \frac{\xi_1}{2}\int_{\mathbb{R}}|\phi_1|^2 d\xi + \frac{\xi_2}{2}\int_{\mathbb{R}}|\phi_2|^2 d\xi, \end{aligned}$$

with constants $\xi_i = \frac{\gamma_i}{d_i}\frac{\sin(\alpha\pi)}{\pi}$.

Lemma 2. *Let* $U = (\varphi, \varphi_t, \psi, \psi_t, \phi_1, \phi_2)$ *be a regular solution of the problem* (6)*. Then, the energy functional defined in* (7) *satisfies the following relation:*

$$\frac{d}{dt}E(t) = -\xi_1\int_{\mathbb{R}}\Big(|\xi|^2+\eta\Big)|\phi_1(\xi,t)|^2\,d\xi - \xi_2\int_{\mathbb{R}}\Big(|\xi|^2+\eta\Big)|\phi_2(\xi,t)|^2\,d\xi.$$

Proof. Multiplying Equations $(6)_1$ and $(6)_3$ by φ_t and ψ_t, respectively, integrating by parts over $(0,L)$, and then summing the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L\Big(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + d_1|\varphi_x+\psi|^2 + d_2|\psi_x|^2\Big)dx\right) - (\varphi_x+\psi)(L)\varphi_t(L) = 0. \tag{8}$$

Multiplying Equations $(6)_2$ and $(6)_4$ by $\xi_1\phi_1$ and $\xi_2\phi_2$, respectively, integrating over $\mathbb{R}$, and adding the resulting equations gives us

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\xi_1\int_{\mathbb{R}}|\phi_1|^2 d\xi + \xi_2\int_{\mathbb{R}}|\phi_2|^2 d\xi\right) + \xi_1\int_{\mathbb{R}}(\xi^2+\eta)|\phi_1(\xi,t)|^2\,d\xi \\ &+ \xi_2\int_{\mathbb{R}}(\xi^2+\eta)|\phi_2(\xi,t)|^2\,d\xi - \xi_1\varphi_t(L)\int_{\mathbb{R}}\mu(\xi)\phi_1(\xi,t)\,d\xi \\ &- \xi_2\psi_t(L)\int_{\mathbb{R}}\mu(\xi)\phi_2(\xi,t)\,d\xi = 0. \end{aligned} \tag{9}$$

Combining Equations (8) and (9), we obtain

$$\frac{d}{dt}E(t) = -\xi_1\int_{\mathbb{R}}\Big(\xi^2+\eta\Big)|\phi_1(\xi,t)|^2\,d\xi - \xi_2\int_{\mathbb{R}}\Big(\xi^2+\eta\Big)|\phi_2(\xi,t)|^2\,d\xi.$$

This concludes the proof of the lemma. □

We now address the well-posedness of (6). To this end, we introduce the following Hilbert space, referred to as the energy space:

$$\mathcal{H} = \left(H_L^1(0,L) \times L^2(0,L)\right)^2 \times L^2(\mathbb{R}),$$

where $H_L^1(0,L)$ is defined as

$$H_L^1(0,L) = \{\varphi \in H^1(0,L) \mid \varphi(0) = 0\}.$$

For $U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2)^T$ and $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4, \tilde{\phi}_1, \tilde{\phi}_2)^T$, we define the inner product in $\mathcal{H}$ as follows:

$$\begin{aligned}\langle U, \tilde{U}\rangle_{\mathcal{H}} &= \int_0^L (\rho_1 u_2 \overline{\tilde{u}_2} + \rho_2 u_4 \overline{\tilde{u}_4}) dx + \int_0^L d_1 (u_{1x} + u_3) \overline{(\tilde{u}_{1x} + \tilde{u}_3)} dx \\ &+ \int_0^L d_2 u_{3x} \overline{\tilde{u}_{3x}} dx + d_1^2 \xi_1 \int_{\mathbb{R}} \phi_1 \overline{\tilde{\phi}_1}\, d\xi + d_2^2 \xi_2 \int_{\mathbb{R}} \phi_2 \overline{\tilde{\phi}_2}\, d\xi.\end{aligned}$$

We transform the system described by (6) into a semigroup framework. By defining the vector function $U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2)^T$, we express the system (6) in the equivalent form

$$\begin{cases} U' = \mathcal{A}U, \quad t > 0, \\ U(0) = U_0, \end{cases}$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \phi_1^0, \phi_2^0)^T$.

The operator $\mathcal{A}$ is linear and defined by

$$\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \frac{d_1}{\rho_1}(u_{1x} + u_3)_x \\ u_4 \\ \frac{d_2}{\rho_2} u_{3xx} - \frac{d_1}{\rho_2}(u_{1x} + u_3) \\ -(\xi^2 + \eta)\phi_1 + u_2(L)\mu(\xi) \\ -(\xi^2 + \eta)\phi_2 + u_4(L)\mu(\xi) \end{pmatrix}.$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u_1, u_2, u_3, u_4, \phi_1, \phi_2)^T \in \mathcal{H} : u_1, u_3 \in H^2 \cap H_L^1, \\ \xi\phi_1, \xi\phi_2 \in L^2(\mathbb{R}), -\left(|\xi|^2 + \eta\right)\phi_1 + u_2(L)\mu(\xi) \in L^2(\mathbb{R}^n), \\ (u_{1x} + u_3)(L) = -\gamma_1 \dfrac{\sin(\alpha\pi)}{\pi} \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(\xi, t) d\xi, \\ -\left(|\xi|^2 + \eta\right)\phi_2 + u_4(L)\mu(\xi) \in L^2(\mathbb{R}), \\ u_{3x}(L) = -\gamma_2 \dfrac{\sin(\alpha\pi)}{\pi} \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(\xi, t) d\xi. \end{array} \right\}. \tag{10}$$

We state the following theorem on existence and uniqueness:

Theorem 2.

1. *If $U_0 \in D(\mathcal{A})$, then the system (6) has a unique strong solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

2. *If $U_0 \in \mathcal{H}$, then the system (6) has a unique weak solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, \mathcal{H}).$$

Remark 1. *Note that, while strong solutions satisfy the differential equation pointwise and require higher regularity, weak solutions are defined in an integral sense with lower regularity requirements.*

Proof. First, we demonstrate that the operator $\mathcal{A}$ is dissipative.

For any $U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2) \in D(\mathcal{A})$, we have

$$\begin{aligned} \mathcal{R}e\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -d_1^2\xi_1 \int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi_1(\xi,t)|^2 d\xi - d_2^2\xi_2 \int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi_2(\xi,t)|^2 d\xi \\ &\leq 0. \end{aligned}$$

Hence, $\mathcal{A}$ is dissipative.

We will show that the operator $I - \mathcal{A}$ is surjective.

Given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, we prove that there exists $U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2) \in D(\mathcal{A})$ satisfying

$$(I - \mathcal{A})U = F.$$

That is,

$$\begin{cases} u_1 - u_2 &= f_1, \\ u_2 - \frac{d_1}{\rho_1}(u_{1x} + u_3)_x &= f_2 \\ u_3 - u_4 &= f_3, \\ u_4 - \frac{d_2}{\rho_2}u_{3xx} + \frac{d_1}{\rho_2}(u_{1x} + u_3) &= f_4 \\ \phi_1(1 + \xi^2 + \eta) - \mu(\xi)u_2(L,t) &= f_5, \\ \phi_2(1 + \xi^2 + \eta) - \mu(\xi)u_4(L,t) &= f_6. \end{cases} \tag{11}$$

Then, $(11)_1$, $(11)_2$, $(11)_5$, and $(11)_6$ yield

$$\begin{cases} u_2 &= u_1 - f_1, \\ u_4 &= u_3 - f_3, \\ \phi_1 &= \frac{f_5}{1+\xi^2+\eta} + \frac{\mu(\xi)u_2(L,t)}{1+\xi^2+\eta}, \\ \phi_2 &= \frac{f_6}{1+\xi^2+\eta} + \frac{\mu(\xi)u_4(L,t)}{1+\xi^2+\eta}. \end{cases} \tag{12}$$

Inserting Equations $(11)_1$ in $(11)_2$ and $(11)_3$ in $(11)_4$, we obtain

$$\begin{cases} \rho_1 u_1 - d_1(u_{1x} + u_3)_x &= \rho_1(f_1 + f_2), \\ \rho_2 u_3 - d_2 u_{3xx} + d_1(u_{1x} + u_3) &= \rho_2(f_3 + f_4). \end{cases} \tag{13}$$

Solving system (13) is equivalent to finding $u_1, u_3 \in H^2(0,L) \cap H^1_L(0,L)$ such that

$$\begin{cases} \int_0^L [\rho_1 u_1 - d_1(u_{1x} + u_3)_x]\chi dx &= \int_0^L \rho_1[f_1 + f_2]\chi dx, \\ \int_0^L [\rho_2 u_3 - d_2 u_{3xx} + d_1(u_{1x} + u_3)]\zeta dx &= \int_0^L \rho_2(f_3 + f_4)\zeta dx, \end{cases} \tag{14}$$

for all $\chi, \zeta \in H^1_L(0,L)$.

Inserting Equations $(12)_3$ in $(14)_1$ and $(12)_4$ in $(14)_2$, we obtain

$$\begin{cases} \int_0^L [\rho_1 u_1 \chi + d_1(u_{1x} + u_3)\chi_x] dx + k_1 u_2(L)\chi(L) \\ = \int_0^L \rho_1[f_1 + f_2]\chi dx - d_1^2\xi_1\chi(L) \int_{\mathbb{R}} \frac{f_5\mu(\xi)}{1+\xi^2+\eta} d\xi, \\ \int_0^L [\rho_2 u_3 \zeta + d_2 u_{3x}\zeta_x + d_1(u_{1x} + u_3)\zeta] dx + k_2 u_4(L)\zeta(L) \\ = \int_0^L \rho_2(f_3 + f_4)\zeta dx - d_2^2\xi_2\zeta(L) \int_{\mathbb{R}} \frac{f_6\mu(\xi)}{1+\xi^2+\eta} d\xi, \end{cases} \tag{15}$$

where $k_i = d_i^2 \xi_i \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta} d\xi$ and with the following boundary conditions:

$$u_2(L) = u_1(L) - f_1(L), \quad u_4(L) = u_3(L) - f_3(L). \tag{16}$$

Inserting (16) into (15), we obtain

$$\begin{cases} \int_0^L [\rho_1 u_1 \chi + d_1(u_{1x} + u_3)\chi_x] dx + k_1 u_1(L)\chi(L) \\ = \int_0^L \rho_1 [f_1 + f_2] \chi dx - d_1^2 \xi_1 \chi(L) \int_{\mathbb{R}} \frac{f_5 \mu(\xi)}{1+\xi^2+\eta} d\xi + k_1 f_1(L)\chi(L), \\ \int_0^L [\rho_2 u_3 \zeta + d_2 u_{3x} \zeta_x + d_1(u_{1x} + u_3)\zeta] dx + k_2 u_3(L)\zeta(L) \\ = \int_0^L \rho_2 (f_3 + f_4)\zeta dx - d_2^2 \xi_2 \zeta(L) \int_{\mathbb{R}} \frac{f_6 \mu(\xi)}{1+\xi^2+\eta} d\xi + k_2 f_3(L)\zeta(L). \end{cases} \tag{17}$$

Thus, the problem (17) can be reformulated as the following problem:

$$a((u_1, u_3), (\chi, \zeta)) = \mathcal{L}(\chi, \zeta), \tag{18}$$

where

$$\begin{aligned} a((u,v),(\chi,\zeta)) &= \int_0^L [\rho_1 u \chi + d_1(u_x + v)(\chi_x + \zeta) + \rho_2 v \zeta + d_2 v_x \zeta_x] dx \\ &+ k_1 u(L)\chi(L) + +k_2 v(L)\zeta(L), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\chi,\zeta) &= \int_0^L \rho_1 [f_1 + f_2] \chi dx - d_1^2 \xi_1 \chi(L) \int_{\mathbb{R}} \frac{f_5 \mu(\xi)}{1+\xi^2+\eta} d\xi + k_1 f_1(L)\chi(L) \\ &+ \int_0^L \rho_2 (f_3 + f_4)\zeta dx - d_2^2 \xi_1 \zeta(L) \int_{\mathbb{R}} \frac{f_6 \mu(\xi)}{1+\xi^2+\eta} d\xi + k_2 f_3(L)\zeta(L). \end{aligned}$$

It is straightforward to verify that a is continuous and coercive and that $\mathcal{L}$ is continuous. By applying the Lax–Milgram Theorem A1, we conclude that, for all $(\chi, \zeta) \in H_L^1(0,L) \times H_L^1(0,L)$, the problem (18) has a unique solution $(u_1, u_3) \in H_L^1(0,L) \times H_L^1(0,L)$.

Using classical elliptic regularity results, it follows from (17) that $(u_1, u_3) \in H^2(0,L) \times H^2(0,L)$. Consequently, the operator $I - \mathcal{A}$ is surjective. Finally, Theorem 2 follows from the Lumer–Phillips Theorem A2. □

3. Asymptotic Stability

In this section, we analyze the asymptotic stability of the system described by (1)–(3), which requires

$$\lim_{t \to +\infty} E(t) = 0, \quad \forall U_0 \in \mathcal{H}.$$

We will examine the spectrum and investigate the strong stability of the C_0 semigroup associated with the system (1)–(3) using the criteria of Arendt–Batty [16].

The main results of this paper are summarized as follows:

Theorem 3. *The semigroup of contractions $(S(t))_{t \geq 0}$ is strongly stable on the energy space $\mathcal{H}$, meaning that*

$$\lim_{t \to \infty} \left\| e^{\mathcal{A}t} U_0 \right\|_{\mathcal{H}} = 0 \quad \forall U_0 \in \mathcal{H}.$$

First, we need to prove the following lemmas:

Lemma 3. *The point spectrum of the operator $\mathcal{A}$ does not intersect with the imaginary axis, i.e.,*

$$\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset,$$

where

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - \mathcal{A}) \neq \{0\}\}.$$

Proof. For clarity, we divide the proof into two steps.

Step 1. By direct computation, the equation

$$\mathcal{A}U = 0$$

with $U \in D(\mathcal{A})$ admits only the trivial solution, i.e., $U = 0$. Hence, $0 \notin \sigma_p(\mathcal{A})$.

Step 2. Suppose that there exists $\beta \in \mathbb{R}^*$ such that

$$\ker(i\beta I - \mathcal{A}) \neq \{0\}.$$

Thus, $\lambda = i\beta$ is an eigenvalue of $\mathcal{A}$. Let U be an eigenvector in $D(\mathcal{A})$ associated with λ, satisfying

$$(i\beta I - \mathcal{A})U = 0.$$

Equivalently, we have

$$\left\{ \begin{array}{lcl} u_2 & = & i\beta u_1, \\ \frac{d_1}{\rho_1}(u_{1x} + u_3)_x & = & i\beta u_2 \\ u_4 & = & i\beta u_3, \\ \frac{d_2}{\rho_2} u_{3xx} - \frac{d_1}{\rho_2}(u_{1x} + u_3) & = & i\beta u_4 \\ -\phi_1(\xi^2 + \eta) + \mu(\xi) u_2(L,t) & = & i\beta\phi_1, \\ -\phi_2(\xi^2 + \eta) + \mu(\xi) u_4(L,t) & = & i\beta\phi_1. \end{array} \right. \tag{19}$$

First, a straightforward computation shows that

$$\begin{array}{rcl} 0 & = & \Re < (i\beta I - \mathcal{A})U, U >_{\mathcal{H}} \\ & = & d_1^2 \xi_1 \int_{\mathbb{R}} \left(\xi^2 + \eta\right) |\phi_1(\xi,t)|^2 d\xi + d_2^2 \xi_2 \int_{\mathbb{R}} \left(\xi^2 + \eta\right) |\phi_2(\xi,t)|^2 d\xi. \end{array}$$

We deduce that $\phi_1 = \phi_2 = 0$ a.e. in $\mathbb{R}$.
On the other hand, by $(19)_5$ and $(19)_6$, we obtain,

$$\phi_j = \frac{\mu(\xi) u_{2j}(L,t)}{\xi^2 + \eta + i\beta},\ j = 1, 2,$$

which yields $u_2(L,t) = u_4(L,t) = 0$. Hence, from $(19)_1$ and $(19)_3$, we obtain

$$u_1(L,t) = u_3(L,t) = 0, \quad \text{and from (10)}, \quad (u_{1x} + u_3)(L,t) = u_{3x}(L,t) = 0. \tag{20}$$

Otherwise, replacing $(19)_1$ into $(19)_2$ and $(19)_3$ into $(19)_4$ and setting $v = u_{1x} + u_3$, we obtain

$$\left\{ \begin{array}{lcl} \beta^2 u_1 + \frac{d_1}{\rho_1} v_x & = & 0, \\ \beta^2 u_3 + \frac{d_2}{\rho_2} u_{3xx} - \frac{d_1}{\rho_2} v & = & 0. \end{array} \right. \tag{21}$$

We can rewrite (21) and (20) as

$$\left\{ \begin{array}{l} \frac{d}{dx} X = \mathcal{B}X, \quad t > 0, \\ X(L) = 0, \end{array} \right.$$

where $X := (u_1, v, u_3, u_{3x})^T$. The operator $\mathcal{B}$ is linear and defined by

$$\mathcal{B}\begin{pmatrix} u_1 \\ v \\ u_3 \\ u_{3x} \end{pmatrix} = \begin{pmatrix} v - u_3 \\ -\frac{\rho_1\beta^2}{d_1}u_1 \\ u_{3x} \\ \frac{d_1}{d_2}v - \frac{\rho_2\beta^2}{d_2}u_3 \end{pmatrix}.$$

According to Picard's theorem for ordinary differential equations, the system (3) has a unique solution, which is $X = 0$. Thus, $u_1 = u_3 = 0$. It follows from (19) that $u_2 = u_4 = 0$. Consequently, we obtain $U = 0$ over the interval $(0, L)$, which contradicts the assumption that $U \neq 0$. □

Lemma 4. *The operator* $\left(i\beta I - \mathcal{A}\right)$ *is surjective.*

Proof. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$ be looking for
$U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2) \in D(\mathcal{A})$ such that

$$i\beta U - \mathcal{A}U = F.$$

That is,

$$\begin{cases} i\beta u_1 - u_2 &= f_1, \\ i\beta u_2 - \frac{d_1}{\rho_1}(u_{1x} + u_3)_x &= f_2 \\ i\beta u_3 - u_4 &= f_3, \\ i\beta u_4 - \frac{d_2}{\rho_2}u_{3xx} + \frac{d_1}{\rho_2}(u_{1x} + u_3) &= f_4 \\ \phi_1(i\beta + \xi^2 + \eta) - \mu(\xi)u_2(L,t) &= f_5, \\ \phi_2(i\beta + \xi^2 + \eta) - \mu(\xi)u_4(L,t) &= f_6, \end{cases}$$

which is equivalent to

$$\begin{cases} u_2 &= i\beta u_1 - f_1, \\ -\beta^2 u_1 - \frac{d_1}{\rho_1}(u_{1x} + u_3)_x &= f_2 + i\beta f_1, \\ u_4 &= i\beta u_3 - f_3, \\ -\beta^2 u_3 - \frac{d_2}{\rho_2}u_{3xx} + \frac{d_1}{\rho_2}(u_{1x} + u_3) &= f_4 + i\beta f_3 \\ \phi_1 &= \frac{f_5 + \mu(\xi)u_2(L,t)}{i\beta + \xi^2 + \eta}, \\ \phi_2 &= \frac{f_6 + \mu(\xi)u_4(L,t)}{i\beta + \xi^2 + \eta}. \end{cases} \tag{22}$$

To solve the last system (22), it is enough to study the following:

$$\begin{cases} \beta^2\rho_1 u_1 + d_1(u_{1x} + u_3)_x &= -\rho_1\left(f_2 + i\beta f_1\right), \\ \beta^2\rho_2 u_3 + d_2 u_{3xx} - d_1(u_{1x} + u_3) &= -\rho_2\left(f_4 + i\beta f_3\right), \end{cases} \tag{23}$$

with the conditions

$$\begin{cases} u_1(0) &= 0, \\ u_3(0) &= 0, \\ (u_{1x} + u_3)(L) &= -d_1\xi_1\Big((f_5 + i\beta u_1(L))I_1(\beta,\eta) - f_1(L)I_2(\beta,\eta)\Big), \\ u_{3x}(L) &= -d_2\xi_2\Big((f_6 + i\beta u_3(L))I_1(\beta,\eta) - f_3(L)I_2(\beta,\eta)\Big), \end{cases}$$

where $I_1(\beta,\eta) = \displaystyle\int_{\mathbb{R}} \frac{\mu(\xi)}{i\beta + \xi^2 + \eta}d\xi$ and $I_2(\beta,\eta) = \displaystyle\int_{\mathbb{R}} \frac{\mu^2(\xi)}{i\beta + \xi^2 + \eta}d\xi$.

We now distinguish two cases.

Step 1. $\beta = 0$ and $\eta > 0$: System (23) is equivalent to finding $u_1, u_3 \in H^2(0,L) \cap H^1_L(0,L)$ such that

$$\left\{ \begin{array}{rcl} -\int_0^L d_1(u_{1x}+u_3)_x \chi dx &=& \int_0^L \rho_1 f_2 \chi dx, \\ \int_0^L [-d_2 u_{3xx} + d_1(u_{1x}+u_3)]\zeta dx &=& \int_0^L \rho_2 f_4 \zeta dx, \end{array} \right. \tag{24}$$

for all $\chi, \zeta \in H^1_L(0,L)$.

Using integration by parts in (24), we deduce that (22) is equivalent to

$$b((u_1,u_3),(\chi,\zeta)) = \mathcal{M}(\chi,\zeta), \tag{25}$$

where

$$b((u,v),(\chi,\zeta)) = \int_0^L [d_1(u_x+v)(\chi_x+\zeta) + d_2 v_x \zeta_x] dx,$$

and

$$\begin{array}{rcl} \mathcal{M}(\chi,\zeta) &=& \int_0^L (\rho_1 f_2 \chi + \rho_2 f_4 \zeta) dx - d_1^2 \xi_1 [f_5 - f_1(L) I_2(0,\eta)]\chi(L) \\ &-& d_2^2 \xi_2 [f_6 - f_3(L) I_2(0,\eta)]\zeta(L). \end{array}$$

It is straightforward to verify that the bilinear form b is continuous and coercive and that the operator $\mathcal{M}$ is continuous. By applying the Lax–Milgram theorem, we conclude that, for all $(\chi,\zeta) \in H^1_L(0,L) \times H^1_L(0,L)$, the problem (25) has a unique solution $(u_1,u_3) \in H^1_L(0,L) \times H^1_L(0,L)$. Utilizing classical elliptic regularity, it follows from (24) that $(u_1,u_3) \in H^2(0,L) \times H^2(0,L)$. Consequently, the operator $-\mathcal{A}$ is surjective.

Step 2. $\beta \neq 0$ and $\eta \geq 0$:

Now, we consider the following system:

$$\left\{ \begin{array}{rcl} -d_1(u_{1x}+u_3)_x &=& g_1, \\ -d_2 u_{3xx} + d_1(u_{1x}+u_3) &=& g_2, \end{array} \right. \tag{26}$$

with the conditions

$$\left\{ \begin{array}{rcl} u_1(0) &=& 0, \\ u_3(0) &=& 0, \\ (u_{1x}+u_3)(L) &=& -i\beta d_1 \xi_1 u_1(L) I_1(\beta,\eta), \\ u_{3x}(L) &=& -i\beta d_2 \xi_2 u_3(L) I_1(\beta,\eta), \end{array} \right.$$

where $(g_1,g_2) \in \left(L^2(0,L)\right)^2$.

Let us note that $\mathcal{L} : (u_1,u_2) \longrightarrow (-d_1(u_{1x}+u_3)_x, -d_2 u_{3xx} + d_1(u_{1x}+u_3))$ with domain $D(\mathcal{L}) = \{(u_1,u_3) \in \left(H^1_L(0,L)\right)^2,\ u_1(0) = 0,\ u_3(0) = 0,\ (u_{1x}+u_3)(L) = -i\beta d_1 \xi_1 u_1(L) I_1(\beta,\eta),$ $u_{3x}(L) = -i\beta d_2 \xi_2 u_3(L) I_1(\beta,\eta)\}$.

Multiplying $(26)_1$ by χ and $(26)_2$ by ζ, one obtains:

$$\begin{array}{rl} & \int_0^L [d_1(u_x+v)(\chi_x+\zeta) + d_2 v_x \zeta_x] dx + i\beta d_1^2 \xi_1 I_1(\beta,\eta) u_1(L)\chi(L) \\ + & i\beta d_2^2 \xi_2 I_2(\beta,\eta) u_3(L)\zeta(L) = \int_0^L (g_1 \chi + g_2 \zeta) dx, \end{array} \tag{27}$$

for all $(\chi,\zeta) \in \left(H^1_L(0,L)\right)^2$.

By applying the Lax–Milgram theorem once more, we deduce that there exists a unique strong solution $(u_1,u_3) \in \left(H^1_L(0,L)\right)^2 \cap D(\mathcal{L})$ for the variational problem (27).

Consequently, it follows that $\mathcal{L}^{-1}$ is compact in $\left(L^2(0,L)\right)^2$ and therefore (23) is equivalent to

$$\left(\beta^2\mathcal{L}^{-1} - I\right)U = \Phi,$$

where $U = (u_1, u_3)$ and $\Phi = \left(-\rho_1(f_2 + i\beta f_1), -\rho_2(f_4 + i\beta f_3)\right)$ and, by Fredholm's alternative, it suffices to prove that $Ker\left(\beta^2\mathcal{L}^{-1} - I\right) = \{0\}$.

For this purpose, let $(y_1, y_2) \in Ker\left(\beta^2\mathcal{L}^{-1} - I\right)$; then, we have

$$\begin{cases} \beta^2\rho_1 y_1 + d_1(y_{1x} + y_3)_x &= 0, \\ \beta^2\rho_2 y_3 + d_2 y_{3xx} - d_1(y_{1x} + y_3) &= 0, \end{cases} \tag{28}$$

with the conditions

$$\begin{cases} y_1(0) &= 0, \\ y_3(0) &= 0, \\ (y_{1x} + y_3)(L) &= -i\beta d_1\xi_1 y_1(L) I_1(\beta,\eta), \\ y_{3x}(L) &= -i\beta d_2\xi_2 y_3(L) I_1(\beta,\eta). \end{cases}$$

Multiplying $(28)_1$ by $\overline{y_1}$ and $(28)_2$ by $\overline{y_3}$, integrating over $(0, L)$, one obtains

$$\begin{aligned} &\int_0^L \left(\beta^2\rho_1|y_1|^2 + \beta^2\rho_2|y_3|^2 - d_2|y_{3x}|^2 - d_1|y_{1x} + y_3|^2\right)dx \\ = \; &-i\beta I_1(\beta,\eta)\left(d_1^2\xi_1|y_1(L)|^2 + d_2^2\xi_2|y_3(L)|^2\right). \end{aligned}$$

Taking the imaginary part, we deduce that

$$d_1^2\xi_1|y_1(L)|^2 + d_2^2\xi_2|y_3(L)|^2 = 0.$$

Hence, we deduce that (y_1, y_2) is the solution of

$$\begin{cases} \beta^2\rho_1 y_1 + d_1(y_{1x} + y_3)_x &= 0, \\ \beta^2\rho_2 y_3 + d_2 y_{3xx} - d_1(y_{1x} + y_3) &= 0, \\ y_1(0) = y_3(0) = (y_{1x} + y_3)(L) &= y_{3x}(L). \end{cases}$$

Using the same argument used in Lemma 3, we infer that $(y_1, y_2) = (0, 0)$.
This completes the proof of Lemma 4. □

From Lemmas 3 and 4, we conclude the following result.

Proposition 1. $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$.

Proof of Theorem 3. Due to Proposition 1, the operator $\mathcal{A}$ lacks pure imaginary eigenvalues, and the intersection $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. By applying the general criterion from Arendt and Batty in [17], the C^0 semigroup $(S(t))_{t\geq 0}$ of contractions is strongly stable. □

4. Lack of Exponential Stability

The primary result of this section is encapsulated in the following theorem.

Theorem 4. *The semigroup generated by the operator $\mathcal{A}$ fails to exhibit exponential stability in the energy space $\mathcal{H}$.*

Proof. Our objective is to demonstrate that an infinite number of eigenvalues of the operator $\mathcal{A}$ approach the imaginary axis, thereby preventing the Timoshenko system (1)–(3)

from achieving exponential stability. To begin, we derive the characteristic equation that determines the eigenvalues of $\mathcal{A}$.

Given that $\mathcal{A}$ is dissipative, we choose a sufficiently small $\alpha_0 > 0$ and examine the asymptotic behavior of the eigenvalues λ of $\mathcal{A}$ within the set $S = \lambda \in \mathbb{C} : \alpha_0 \leq \mathrm{Re}(\lambda) \leq 0$.

We first establish the characteristic equation that the eigenvalues of $\mathcal{A}$ must satisfy. Let $\lambda \in \mathbb{C}^*$ be an eigenvalue of $\mathcal{A}$ and let $U = (u_1, \lambda u_1, u_3, \lambda u_3, \phi_1, \phi_2) \in D(\mathcal{A})$ be a corresponding eigenvector such that $|U| = 1$.

The resulting eigenvalue problem is then given by

$$\begin{cases} \lambda^2 u_1 - \frac{d_1}{\rho_1} u_{1xx} - \frac{d_1}{\rho_1} u_{3x} &= 0, \\ \lambda^2 u_3 - \frac{d_2}{\rho_2} u_{3xx} + \frac{d_1}{\rho_2} u_{1x} + \frac{d_1}{\rho_2} u_3 &= 0, \\ \phi_1 = \frac{\mu(\xi) u_1(L,t)}{\xi^2+\eta+\lambda}, \quad \phi_2 = \frac{\mu(\xi) u_3(L,t)}{\xi^2+\eta+\lambda}, \\ u_1(0) = u_3(0) = 0, \ \ (u_{1x} + u_3)(L) &= -d_1 \xi_1 u_1(L) I_\alpha(\lambda, \eta) \\ u_{3x}(L) &= -d_2 \xi_2 u_3(L) I_\alpha(\lambda, \eta). \end{cases}$$

where $I_\alpha(\lambda, \eta) = \displaystyle\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\xi^2 + \eta + \lambda} d\xi$.

Equivalently, we have

$$\begin{cases} u_{1xxxx} - \lambda^2 \left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right) u_{1xx} + \lambda^2 \frac{\rho_1 \rho_2}{d_1 d_2} \left(\lambda^2 + \frac{d_1}{\rho_2}\right) u_1 &= 0, \\ u_1(0) = u_3(0) &= 0, \\ \left(\frac{\rho_1}{d_1} \lambda^2 - \gamma_1 \gamma_2 (\lambda + \eta)^{2\alpha-2}\right) u_1(L) - \gamma_2 (\lambda + \eta)^{\alpha-1} u_{1x}(L) - u_{1xx}(L) &= 0, \\ \frac{\rho_2 \gamma_1}{d_2} \left(\lambda^2 + \frac{d_1}{\rho_2}\right) u_1(L)(\lambda + \eta)^{\alpha-1} + \left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right) \lambda^2 u_{1x}(L) - u_{1xxx}(L) &= 0, \end{cases} \tag{29}$$

where we used $I_\alpha(\lambda, \eta) = \dfrac{\gamma_i}{d_i \xi_i} (\lambda + \eta)^{\alpha-1}$, $i = 1, 2$ (see Lemma 2.1 in [6] for the proof).

The characteristic polynomial associated with System (29) is given by

$$P(r) := r^4 - \lambda^2 \left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right) r^2 + \lambda^2 \frac{\rho_1 \rho_2}{d_1 d_2} \left(\lambda^2 + \frac{d_1}{\rho_2}\right) = 0.$$

Our goal is to analyze the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ within the set S. A detailed examination reveals that the polynomial P has four distinct roots when $\left(\frac{\rho_1}{d_1} - \frac{\rho_2}{d_2}\right)^2 \lambda^2 \neq 4 \frac{\rho_1}{d_2}$.

Thus, the four distinct roots of P are given by $r_1(\lambda)$, $r_2(\lambda)$, $r_3(\lambda) = -r_1(\lambda)$, and $r_4(\lambda) = -r_2(\lambda)$, where

$$\begin{cases} r_1(\lambda) = \frac{\lambda}{\sqrt{2}} \sqrt{\left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right) + \sqrt{\left(\frac{\rho_1}{d_1} - \frac{\rho_2}{d_2}\right)^2 - 4 \frac{\rho_1}{d_2} \lambda^{-2}}}, \\ r_2(\lambda) = \frac{\lambda}{\sqrt{2}} \sqrt{\left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right) - \sqrt{\left(\frac{\rho_1}{d_1} - \frac{\rho_2}{d_2}\right)^2 - 4 \frac{\rho_1}{d_2} \lambda^{-2}}}. \end{cases}$$

The general solution to (29) can be expressed as

$$u_1(x) = c_1 \sinh(r_1(\lambda) x) + c_2 \sinh(r_2(\lambda) x) + c_3 \cosh(r_1(\lambda) x) + c_4 \cosh(r_2(\lambda) x).$$

Applying the boundary conditions in (29) at $x = 0$ yields $c_3 = c_4 = 0$. Additionally, the boundary conditions at $x = L$ in (29) can be expressed as

$$M(\lambda)C = \begin{pmatrix} f_1(r_1)\sinh(r_1L) - f_2(r_1)\cosh(r_1L) & f_1(r_2)\sinh(r_2L) - f_2(r_2)\cosh(r_2L) \\ H_1\sinh(r_1L) + g(r_1)\cosh(r_1L) & H_1\sinh(r_2L) + g(r_2)\cosh(r_2L) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{array}{l} f_1(r) = \frac{\rho_1}{d_1}\lambda^2 - \gamma_1\gamma_2(\lambda+\eta)^{2\alpha-2} - r^2, \quad f_2(r) = \gamma_2\lambda(\lambda+\eta)^{\alpha-1}r, \\ H_1 = \frac{\rho_2\gamma_1}{d_2}(\lambda^2 + \frac{d_1}{\rho_2})(\lambda+\eta)^{\alpha-1}, \quad g(r) = \left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right)\lambda^2 r - r^3 \text{ and } C = (c_1\, c_2)^T. \end{array} \tag{30}$$

Let $\det(M)$ represent the determinant of the matrix M. Then, it follows that

$$\begin{array}{rcl} det M(\lambda) & = & (f_1(r_1) - f_1(r_2))H_1\sinh(r_1L)\sinh(r_2L) \\ & - & (f_2(r_1)g(r_2) - f_2(r_2)g(r_1))\cosh(r_1L)\cosh(r_2L) \\ & + & (f_1(r_1)g(r_2) + f_2(r_2)H_1)\sinh(r_1L)\cosh(r_2L) \\ & - & (f_2(r_1)H_1 + f_1(r_2)g(r_1))\cosh(r_1L)\sinh(r_2L). \end{array}$$

The Equation (29) has a non-trivial solution if and only if $\det(M) = 0$.
Case 1. Assuming that $\frac{\rho_1}{d_1} = \frac{\rho_2}{d_2}$, and applying the asymptotic expansion, we obtain

$$\begin{array}{l} r_1 = \lambda\sqrt{\frac{\rho_1}{d_1}}\sqrt{1 + i\frac{d_1}{\sqrt{\rho_1 d_2}}\frac{1}{\lambda}} = \sqrt{\frac{\rho_1}{d_1}}\lambda + \frac{i}{2}\sqrt{\frac{d_1}{d_2}} + \frac{1}{8}\frac{d_1^{3/2}}{\sqrt{\rho_1}d_2}\frac{1}{\lambda} + o(\frac{1}{\lambda}), \\ r_2 = \lambda\sqrt{\frac{\rho_1}{d_1}}\sqrt{1 - i\frac{d_1}{\sqrt{\rho_1 d_2}}\frac{1}{\lambda}} = \sqrt{\frac{\rho_1}{d_1}}\lambda - \frac{i}{2}\sqrt{\frac{d_1}{d_2}} + \frac{1}{8}\frac{d_1^{3/2}}{\sqrt{\rho_1}d_2}\frac{1}{\lambda} + o(\frac{1}{\lambda}). \end{array} \tag{31}$$

Next, inserting (31) into (30), we obtain

$$\begin{cases} (f_1(r_1) - f_1(r_2))H_1 = -2i\gamma_1\sqrt{\frac{\rho_1}{d_2}}\lambda^{2+\alpha} + O(\lambda^{1+\alpha}), \\ f_2(r_1)g(r_2) - f_2(r_2)g(r_1) = 2i\gamma_2\sqrt{\frac{\rho_1}{d_2}}\lambda^{3+\alpha} + O(\lambda^{2+\alpha}), \\ f_1(r_1)g(r_2) + f_2(r_2)H_1 = -i\sqrt{\frac{\rho_1}{d_2}}\lambda^3 + O(\lambda^2), \\ f_2(r_1)H_1 + f_1(r_2)g(r_1) = i\gamma_1\gamma_2\frac{\rho_1}{d_2}\sqrt{\frac{\rho_1}{d_2}}\lambda^{2+2\alpha} + O(\lambda^{1+2\alpha}). \end{cases} \tag{32}$$

where we used

$$(\lambda+\eta)^{\alpha-1} = \lambda^{\alpha-1} + O\left(\lambda^{\alpha-2}\right).$$

From (31), we obtain

$$\begin{cases} \sinh(r_1L) = \sinh\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) + O(\frac{1}{\lambda}) & \cosh(r_1L) = \cosh\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) + O(\frac{1}{\lambda}), \\ \sinh(r_2L) = \sinh\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) + O(\frac{1}{\lambda}) & \cosh(r_2L) = \cosh\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) + O(\frac{1}{\lambda}). \end{cases} \tag{33}$$

Therefore, from (32) and (33), we obtain

$$\begin{array}{rcl} \frac{\det M}{\lambda^{3+\alpha}} & = & 2i\gamma_2\sqrt{\frac{\rho_1}{d_2}}\cosh^2\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) - i\sqrt{\frac{\rho_1}{d_2}}\frac{1}{\lambda^\alpha}\frac{1}{2}\sinh\left(2\sqrt{\frac{\rho_1}{d_1}}\lambda L\right) \\ & - & i\gamma_1\gamma_2\frac{\rho_1}{d_2}\sqrt{\frac{\rho_1}{d_2}}\frac{1}{\lambda^{1-\alpha}}\frac{1}{2}\sinh\left(2\sqrt{\frac{\rho_2}{d_2}}\lambda L\right) + O\left(\lambda^{-1}\right). \end{array} \tag{34}$$

Let λ be a large eigenvalue of $\mathcal{A}$. Then, according to (34), λ is an approximate root of the following asymptotic equation:

$$\varphi(\lambda) = \varphi_0(\lambda) + \frac{\varphi_1(\lambda)}{\lambda^{\min(\alpha,1-\alpha)}} + O(\lambda^{-1}),$$

where $\varphi_0(\lambda) = 2i\gamma_2\sqrt{\frac{\rho_1}{d_2}}\cosh^2\left(\sqrt{\frac{\rho_1}{d_1}}\lambda L\right)$ and $\varphi_1(\lambda) = c\sinh\left(2\sqrt{\frac{\rho_1}{d_1}}\lambda L\right)$.

It is important to note that φ_0 and φ_1 remain bounded within the strip $\alpha_0 \leq \mathrm{Re}(\lambda) \leq 0$.

The roots of φ_0 are given by $i\frac{(2n+1)\pi}{2L}\sqrt{\frac{d_2}{\rho_1}}$, $k \in \mathbb{Z}$, and we conclude using Rouché's theorem.

Case 2. $\frac{\rho_1}{d_1} \neq \frac{\rho_2}{d_2}$ is treated in a similar way.

The proof of Theorem 4 is thus concluded. □

5. The Rate of Decay of the $\mathcal{C}_0$ Semigroup

This section focuses on analyzing the asymptotic behavior of the solution to the system (1)–(3). We demonstrate the polynomial stability of the system (1)–(3):

Theorem 5. *Let $(S(t))_{t\geq 0}$ be the bounded C_0 semigroup on the Hilbert space $\mathcal{H}$; with generator $\mathcal{A}$, we have*

$$\|S(t)\mathcal{A}(I-\mathcal{A})^{-1}\| = O(t^{-1}), \quad t \to \infty.$$

The following corollary follows from Theorem 5 and Remark 8.5 in [16].

Corollary 1. *Given $(u_0, u_1) \in D(\mathcal{A}) \cap R(\mathcal{A})$. There exist constants C, $t_0 > 0$ such that, for all $t \geq t_0$,*

$$\|S(t)(u_0,u_1)\| \leq \frac{C}{t}\|(u_0,u_1)\|_{D(\mathcal{A})\cap R(\mathcal{A})}.$$

To establish Theorem 5, we derive a specific resolvent estimate using a result from Batty, Chill, and Tomilov as presented in [16]. More precisely, we have the following lemmas:

Lemma 5. *The operator $\mathcal{A}$ defined by (2) and (10) satisfies*

$$\limsup_{\beta\in\mathbb{R},|\beta|\to+\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. By contradiction, suppose that

$$\limsup_{\beta\in\mathbb{R},|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

There exists a sequence of real numbers $\beta_n > 0$ with $\beta_n \to \infty$ and a sequence of vectors $(U_n)_n \in D(\mathcal{A})$ such that

$$\|U_n\|_{\mathcal{H}} = 1,$$

and

$$(i\beta_n I - \mathcal{A})U_n =: F_n = o(1) \text{ in } \mathcal{H}. \tag{35}$$

Our objective is to show that U_n converges to zero, leading to a contradiction.

Note that (35) is equivalent to

$$\begin{cases} i\beta_n u_1^n - u_2^n &= f_1^n \longrightarrow 0 \text{ in } H_L^1, \\ i\beta_n u_2^n - \frac{d_1}{\rho_1}\left(u_{1x}^n + u_3^n\right)_x &= f_2^n \longrightarrow 0 \text{ in } L^2, \\ i\beta_n u_3^n - u_4^n &= f_3^n \longrightarrow 0 \text{ in } H_L^1, \\ i\beta_n u_4^n - \frac{d_2}{\rho_2}u_{3xx}^n + \frac{d_1}{\rho_2}\left(u_{1x}^n + u_3^n\right) &= f_4^n \longrightarrow 0 \text{ in } L^2, \\ \phi_1^n(i\beta_n + \xi^2 + \eta) - \mu(\xi)u_2^n(L,t) &= f_5^n \longrightarrow 0 \text{ in } L^2, \\ \phi_2^n(i\beta_n + \xi^2 + \eta) - \mu(\xi)u_4^n(L,t) &= f_6^n \longrightarrow 0 \text{ in } L^2, \\ (u_{1x}^n + u_3^n)(L) &= -\gamma_1 \frac{\sin(\alpha\pi)}{\pi}\int_{\mathbb{R}} \mu(\xi)\phi_1^n(\xi,t)d\xi, \\ u_{3x}^n(L) &= -\gamma_2 \frac{\sin(\alpha\pi)}{\pi}\int_{\mathbb{R}} \mu(\xi)\phi_2^n(\xi,t)d\xi. \end{cases} \tag{36}$$

First, taking the real part of the inner product of (35) with U_n in $\mathcal{H}$, we obtain

$$\begin{aligned} &\mathcal{R}e\langle i\beta_n I - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}} \\ =\ & d_1^2\xi_1 \int_{\mathbb{R}} \left(\xi^2 + \eta\right)|\phi_1^n(\xi,t)|^2 d\xi + d_2^2\xi_2 \int_{\mathbb{R}} \left(\xi^2 + \eta\right)|\phi_2^n(\xi,t)|^2 d\xi. \end{aligned} \tag{37}$$

Then, from (35) and (37), we obtain

$$\|\phi_1^n\|_{L^2} = \|\phi_2^n\|_{L^2} = o(1),$$

and we deduce that

$$(u_{1x}^n + u_3^n)(L) = o(1) \text{ and } u_{3x}^n(L) = o(1).$$

Note also that we deduce from $(36)_1$ and $(36)_3$ that $\|u_1^n\|_{L^2} = o(1)$ and $\|u_3^n\|_{L^2} = o(1)$. Now, inserting $(36)_1$ into $(36)_2$ and $(36)_3$ into $(36)_4$, we obtain

$$\begin{cases} u_2^n &= i\beta u_1^n - f_1^n, \\ -\beta^2 u_1^n - \frac{d_1}{\rho_1}(u_{1x}^n + u_3^n)_x &= f_2^n + i\beta f_1^n, \\ u_4^n &= i\beta u_3^n - f_3^n, \\ -\beta^2 u_3^n - \frac{d_2}{\rho_2}u_{3xx}^n + \frac{d_1}{\rho_2}(u_{1x}^n + u_3^n) &= f_4^n + i\beta f_3^n \\ \phi_1^n &= \frac{f_5^n + \mu(\xi)u_2^n(L,t)}{i\beta + \xi^2 + \eta}, \\ \phi_2^n &= \frac{f_6^n + \mu(\xi)u_4^n(L,t)}{i\beta + \xi^2 + \eta}. \end{cases} \tag{38}$$

We will break the proof into several steps, and, for simplicity, we will omit the index n.

Step 1. Multiplying $(38)_2$ by $x\overline{u_{1x}}$ and integrating over $(0, L)$, one obtains

$$\begin{aligned} & \beta^2 \int_0^L \frac{|u_1|^2}{2}dx - \frac{\beta^2 L}{2}|u_1(L)|^2 + \frac{d_1 L}{\rho_1}\int_0^L \frac{|u_{1x}|^2}{2}dx - \frac{d_1}{\rho_1}|u_{1x}(L)|^2 \\ - & \frac{d_1}{\rho_1}\int_0^L x\Re(u_{3x}\overline{u_{1x}})dx \\ = & \Re\Big[\int_0^L x f_2 \overline{u_{1x}}dx - i\beta\int_0^L (f_1 + x f_{1x})\overline{u_1}dx + i\beta L f_1(L)\overline{u_1(L)}\Big]. \end{aligned}$$

Using the fact that $\int_0^L x f_2\overline{u_{1x}}dx = o(1)$, $\int_0^L (f_1 + x f_{1x})\overline{\beta u_1}dx = o(1)$ and $|f_1(L)| = |\int_0^L f_{1x}dx| \leq \sqrt{L}\|f_1\|_{H_L^1(0,L)} = o(1)$.

Therefore,

$$\begin{aligned} & \beta^2 \int_0^L \frac{|u_1|^2}{2}dx - \frac{\beta^2 L}{2}|u_1(L)|^2 + \frac{d_1}{\rho_1}\int_0^L \frac{|u_{1x}|^2}{2}dx - \frac{d_1 L}{\rho_1}|u_{1x}(L)|^2 \\ - & \frac{d_1}{\rho_1}\int_0^L x\Re(u_{3x}\overline{u_{1x}})dx = o(1)\Big(1+\Re(\beta\overline{u_1(L)}\Big). \end{aligned} \tag{39}$$

Analogously, by multiplying Equation (38)$_4$ by $x\overline{u_{3x}}$, we obtain

$$\begin{aligned} & (\beta^2 - \frac{d_1}{\rho_2})\int_0^L \frac{|u_3|^2}{2}dx + L(\frac{d_1}{\rho_2} - \beta^2)\frac{|u_3(L)|^2}{2} + \frac{d_2}{\rho_2}\int_0^L \frac{|u_{3x}|^2}{2}dx - \frac{d_2 L|u_{3x}|^2}{2\rho_2} \\ + & \frac{d_1}{\rho_2}\int_0^L x\Re(u_{1x}\overline{u_{3x}})dx = o(1)\Big(1+\Re(\beta\overline{u_3(L)}\Big). \end{aligned} \tag{40}$$

Consequently, estimates (39) and (40) give

$$\begin{aligned} & \beta^2\rho_1 \int_0^L \frac{|u_1|^2}{2}dx - \frac{\beta^2\rho_1 L}{2}|u_1(L)|^2 + d_1\int_0^L \frac{|u_{1x}|^2}{2}dx - d_1 L|u_{1x}(L)|^2 \\ + & (\beta^2\rho_2 - d_1)\int_0^L \frac{|u_3|^2}{2}dx + L(d_1 - \beta^2\rho_2)\frac{|u_3(L)|^2}{2} + d_2\int_0^L \frac{|u_{3x}|^2}{2}dx \\ = & o(1)\Big(1+\Re(\beta\overline{u_1(L)} + \Re(\beta\overline{u_3(L)}\Big). \end{aligned} \tag{41}$$

Step 2. From (38)$_5$ and (38)$_6$, we have

$$\begin{cases} \phi_1^2 &= \frac{f_5^2}{\beta^2+(\xi^2+\eta)^2} + \frac{\mu^2(\xi)u_2^2(L,t)}{\beta^2+(\xi^2+\eta)^2} + 2\mathcal{R}e\frac{f_5\mu(\xi)u_2(L,t)}{\beta^2+(\xi^2+\eta)^2}, \\ \phi_2^2 &= \frac{f_6^2}{\beta^2+(\xi^2+\eta)^2} + \frac{\mu^2(\xi)u_4^2(L,t)}{\beta^2+(\xi^2+\eta)^2} + 2\mathcal{R}e\frac{f_6\mu(\xi)u_4(L,t)}{\beta^2+(\xi^2+\eta)^2}. \end{cases}$$

Given that $\|\phi_1\|_{L^2} = \|\phi_2\|_{L^2} = o(1)$ and $\int_{\mathbb{R}} \frac{\mu(\xi)}{\beta^2+(\xi^2+\eta)^2}\,d\xi > 0$, we can deduce that

$$u_2(L) \longrightarrow 0 \text{ and } u_4(L) \longrightarrow 0.$$

Consequently, using (38)$_1$ (38)$_3$, and the fact that $|f_i(L)| \le \sqrt{L}\|f_i\|_{H_L^1} = O(1)$, $i = 1,3$, we obtain

$$\beta u_1(L) \longrightarrow 0 \text{ and } \beta u_3(L) \longrightarrow 0,$$

and, next, $u_{1x}(L) = o(1)$.

Combining this with (41), (38)$_1$, and (38)$_3$, we obtain that $\|u_2\|_{L^2} = o(1)$ and $\|u_4\|_{L^2} = o(1)$ and, next, $\|U\|_{\mathcal{H}} = o(1)$, which contradicts the hypothesis that $\|U\|_{\mathcal{H}} = 1$.

Thus, the proof of the lemma is complete. □

Lemma 6. *The operator $\mathcal{A}$ defined by* (2) *and* (10) *satisfies*

$$\limsup_{\beta\in\mathbb{R},\,\beta\to 0} \|\beta\big(i\beta I - \mathcal{A}\big)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. By contradiction, suppose that

$$\limsup_{\beta\in\mathbb{R},\,\beta\to 0} \|\beta\big(i\beta I - \mathcal{A}\big)^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty. \tag{42}$$

Put $\beta = \dfrac{1}{\gamma}$ so (42) is equivalent to

$$\limsup_{\gamma\in\mathbb{R},\,|\gamma|\to+\infty} \|\gamma^{-1}(i\gamma^{-1}I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Then, there exists a sequence of real numbers $\gamma_n > 1$ with $\gamma_n \to \infty$ and a sequence of functions $(U_n)_n \in D(\mathcal{A})$ such that

$$\|U_n\|_{\mathcal{H}} = 1, \tag{43}$$

and

$$\gamma_n(i\gamma_n^{-1}I - \mathcal{A})U_n =: F_n = o(1), \quad \text{in } \mathcal{H}. \tag{44}$$

We will demonstrate that $\|U_n\|_{\mathcal{H}} = o(1)$, which contradicts the assumption regarding U_n. To simplify, we will drop the index n in the following and divide the remainder of the proof into two steps for clarity.

Step 1. In fact, (44) can be written as

$$\begin{cases} iu_1 - \gamma u_2 = f_1 & \longrightarrow 0 \quad in \quad H_L^1, \\ iu_2 - \frac{\gamma d_1}{\rho_1}(u_{1x} + u_3)_x = f_2 & \longrightarrow 0 \quad in \quad L^2, \\ iu_3 - \gamma u_4 = f_3 & \longrightarrow 0 \quad in \quad H_L^1, \\ iu_4 - \frac{\gamma d_2}{\rho_2}u_{3xx} + \frac{\gamma d_1}{\rho_2}(u_{1x} + u_3) = f_4 & \longrightarrow 0 \quad in \quad L^2, \\ \phi_1(i + \gamma\xi^2 + \gamma\eta) - \gamma\mu(\xi)u_2(L,t) = f_5 & \longrightarrow 0 \quad in \quad L^2, \\ \phi_2(i + \gamma\xi^2 + \gamma\eta) - \gamma\mu(\xi)u_4(L,t) = f_6 & \longrightarrow 0 \quad in \quad L^2. \end{cases} \tag{45}$$

Since U_n is bounded by 1 in $\mathcal{H}$ and F_n converges to 0 in $\mathcal{H}$, $(45)_1$ and $(45)_3$ imply that

$$\|u_2\|_{L^2} = \|u_4\|_{L^2} = o(1). \tag{46}$$

By taking the real part of the inner product of (44) with U in $\mathcal{H}$, we obtain

$$\begin{aligned} \mathcal{R}e\Big\langle \gamma(i\gamma^{-1}I - \mathcal{A})U, U \Big\rangle_{\mathcal{H}} &= \gamma d_1^2 \xi_1 \int_{\mathbb{R}} \Big(\xi^2 + \eta\Big)|\phi_1^n(\xi,t)|^2 d\xi \\ &+ \gamma d_2^2 \xi_2 \int_{\mathbb{R}} \Big(\xi^2 + \eta\Big)|\phi_2^n(\xi,t)|^2 d\xi. \end{aligned}$$

Then, from (43) and (44), we obtain

$$\|\phi_1\|_{L^2} = \|\phi_2\|_{L^2} = o(1). \tag{47}$$

Considering that $U \in D(\mathcal{A})$, we deduce that

$$(u_{1x} + u_3)(L) = o(1) \text{ and } u_{3x}(L) = o(1). \tag{48}$$

Next, we use the same steps used in the previous lemma to obtain

$$\gamma u_2(L) \longrightarrow 0 \text{ and } \gamma u_4(L) \longrightarrow 0,$$

and we infer that

$$u_1(L) \longrightarrow 0 \text{ and } u_3(L) \longrightarrow 0. \tag{49}$$

Step 2. Now, from $(45)_2$, $(45)_4$, and (46), we obtain

$$\begin{cases} \gamma(u_{1x} + u_3)_x &= -\frac{\rho_1}{d_1}(f_2 - iu_2) := g_1 \longrightarrow 0 \text{ in } L^2, \\ \gamma u_{3xx} - \frac{\gamma d_1}{d_2}(u_{1x} + u_3) &= -\frac{\rho_2}{d_2}(f_4 - iu_4) := g_2 \longrightarrow 0 \text{ in } L^2. \end{cases}$$

Consequently, $V := (u_1, u_{1x} + u_3, u_3, u_{3x})^T$ is the solution of

$$\begin{cases} \frac{d}{dx}V &= CV + G \\ V(L) &= (u_1(L), (u_{1x} + u_3)(L), u_3(L), u_{3x}(L))^T. \end{cases}$$

Here, C is defined as

$$C = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_1}{d_2} & 0 & 0 \end{pmatrix}$$

and $G := (0, \frac{1}{\gamma} g_1, 0, \frac{1}{\gamma} g_2)^T$.

On one hand, by Duhamel's formula, one obtains

$$V(x) = \exp\big(C(x-L)\big)V(L) + \int_L^x \exp\big(C(x-s)\big)G(s)ds.$$

Conversely, a straightforward calculation yields the characteristic polynomial of C:

$$P(\lambda) = \lambda^4.$$

Hence, we obtain

$$\exp(Cx) = \begin{pmatrix} 1 & x - \frac{d_1 x^3}{6d_2} & -x & -\frac{x^2}{2} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{d_1 x^2}{2d_2} & 1 & x \\ 0 & \frac{d_1 x}{d_2} & 0 & 0 \end{pmatrix}, \ \forall x \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} u_1 &= u_1(L) + \Big(x - L - \frac{d_1(x-L)^3}{6d_2}\Big)(u_{1x} + u_3)(L) - (x-L)u_3(L) \\ &- \frac{(x-L)^2}{2} u_{3x}(L) + \frac{1}{\gamma}\int_L^x \Big((x - s - \frac{d_1(x-s)^3}{6d_2})g_1 - \frac{(x-s)^2}{2} g_2\Big)ds, \end{aligned}$$

and, from (48) and (49), it is simple to show that u_1 converges to 0 in L^2. We also show that u_{1x}, u_3 and u_{3x} converge to 0 in L^2.

Finally, combining the last result with (46) and (47), we find that $\|U\|_{\mathcal{H}} = o(1)$, which contradicts the assumption that $\|U\|_{\mathcal{H}} = 1$. □

Proof of Theorem 5. It follows immediately from Lemma 5, Lemma 6, and Theorem 7.6 in [16] that

$$\|S(t)\mathcal{A}(I-\mathcal{A})^{-1}\| = O(t^{-1}), \quad t \to \infty.$$

□

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Appendix A

In this appendix, we present several key theorems from the literature that have been utilized in various proofs throughout this article. We begin with the Lax–Milgram theorem, which serves as a type of representation theorem for bounded linear functionals on a Hilbert space $\mathcal{H}$:

Theorem A1. *Let B be a bounded, coercive bilinear form on a Hilbert space $\mathcal{H}$. Then, for every bounded linear functional ℓ on $\mathcal{H}$, there exists a unique element $x_\ell \in \mathcal{H}$ such that $\ell(x) = B(x, x_\ell)$ for all x in $\mathcal{H}$.*

We then proceed to the Lumer–Phillips theorem, which provides a necessary and sufficient condition for a linear operator in a Banach space to generate a contraction semigroup:

Theorem A2. *Let $\mathcal{A}$ be a densely defined operator on X. Then, $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions on X if and only if*

1. *$\mathcal{A}$ is dissipative;*
2. *$(I - \mathcal{A})D(\mathcal{A}) = X$, where $D(\mathcal{A})$ is the domain of the operator $\mathcal{A}$.*

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