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Research article

Global solution for wave equation involving the fractional Laplacian with logarithmic nonlinearity

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Abstract: We construct the global existence for a wave equation involving the fractional Laplacian with a logarithmic nonlinear source by using the Galerkin approximations. The corresponding results for equations with classical Laplacian are considered as particular cases of our assertions.

Keywords: fractional Laplacian; differential equations; global existence; partial differential equations; logarithmic nonlinearity; Galerkin approximations

1. Introduction

It is well known that the fractional integro-differentiation operation can be considered as an extension of the differentiation operations. It is well known that the idea of fractional differentiation as an extension of the concept of derivatives to the non-integer value arose almost together with the concept of differentiation. The first mention of this idea appears in the correspondence of G. W. Leibniz and the Marquis de l'Hospital in 1695; see [1]. It was then developed by L. Euler, where the expression gives meaning even for non-integer values. The explicit advanced calculation was given in many references. If one replaces the classical Laplacian operator by fractional Laplacian, it will be motivated by the need to represent anomalous waves. The main mathematical models are the fractional Laplacians that have special symmetry and in-variance properties.

The basic evolution equation is

$$u_{tt} + (-\Delta)^s u = 0, s \in (0, 1).$$

In principl, the fractional wave is linear, in which s is some interpolation power of the Laplacian and one can conduct harmonic analysis, then one can generate the semi-groups. The researchers who performed the analysis were not inclined to analysis; the evolution associated with the fractional operator was carried out in stochastic processes because it was discovered that the typical approach to Brownian motion, this type of equations, was not a relevant model for many processes where there is a lack of convergence. Intense work in stochastic processes for several decades, but not in PDE analysis until tens years ago, initiated around pr. Caffarelli but only in the linear case in which we can return to the nonlinear case, while forgetting the probabilities.

In recent years, fractional Laplacian operators and related equations have an increasingly wide utilization in many important fields. In connection with the intensive development of industry, the electric power industry, the theory of nonlinear oscillations, automatic control, and optimal processes, the theory of damped partial differential equations is being developed, and its methods are actively used to solve problems in various fields of natural science and technology, especially when it comes with the fractional Laplacian. For more recent results involving the fractional Laplacian, for example [2–8] and the references therein. Recently, the fractional hyperbolic problems with continuous non-linearities have been studied by many researchers. For example, the authors in [5] studied the initial-boundary value problem of degenerate Kirchhoff-type for $y \in \Gamma_1, t \in \mathbb{R}_+$

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta)^r w = |w|^{p-1} w, \tag{1.1}$$

where $\Gamma_1 \subset \mathbb{R}^n, 1 \le n$ is a bounded domain with Lipshcitz boundary $\partial\Gamma_1$, $[w]$ is the Gagliardo semi-norm of w, $r \in (0, 1), 1 \le \theta < \frac{2_r^\star}{2}, 2_r^\star = \frac{2n}{(n-2r)}, p \in (2\theta - 1, 2_r^\star - 1]$ and $[w]_r$ is the Gagliardo semi-norm of w defined by

$$[w]_r = \sqrt{\int_\Gamma \int_\Gamma \frac{w(y) - w(z)|^2}{|y - z|^{n+2r}} dydz}.$$

By using the Galerkin method, the global existence/nonexistence is obtained for solutions of (1.1) under certain conditions. Furthermore, in the work [9], it is proposed the following damped equation for $y \in \Gamma, t \in \mathbb{R}_+$

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta_y)^r w + |\partial_t w|^{\alpha-1}\partial_t w + w = |w|^{p-2} w, \tag{1.2}$$

where $2 < \alpha < 2\theta < p < 2^\star < r$. Under some natural assumptions, the authors obtained the global existence, vacuum isolating, asymptotic behavior, and blow-up of solutions for (1.2) by combining the Galerkin method with potential wells theory (see [10–15]). In [9], Lin et al. studied the initial-boundary value problem of the Kirchhoff wave equation for $y \in \Gamma, t \in \mathbb{R}_+$

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta_y)^r w = |w|^{p-2} w. \tag{1.3}$$

Regarding the Galerkin method's explanation with related logarithmic nonlinearities, we can review [16–19].

In the present paper, we consider IBVP involving the fractional Laplacian non-linearity. To begin with, let $w = w(y,t)$, $\Gamma_1 \subset \mathbb{R}^n, n \geq 1$ with Lipschitz boundary $\partial\Gamma_1$ and $\Gamma_2 = \mathbb{R}^n \setminus \Gamma_1, t > 0$

$$\begin{cases} \partial_t^2 w + (-\Delta_y)^r w + (-\Delta_y)^r \partial_t w = w\,|w|^{p-2}\log(|w|) & y \in \Gamma_1 \\ w = 0, & y \in \Gamma_2 \\ w(y, t=0) = w_0(y), \partial_t w(y, t=0) = w_1(y) & y \in \Gamma_1, \end{cases} \tag{1.4}$$

where $r \in (0,1)$. The parameter p satisfies

$$2 < p < \frac{2n}{n-2r} = 2_r^\star,\ n > 2r. \tag{1.5}$$

The rest of the paper is organized as follows. In Sections 2 and 3, we introduce our problem and recall necessary definitions and properties of the fractional Sobolev spaces. In Section 4, we study the global existence of solutions for our main problem (1.4).

2. Auxiliary results and function spaces

In this section, we first recall some necessary definitions and properties of the fractional Sobolev spaces, see [20].

The fractional Laplacian of order r, $(-\Delta_y)^r$ of the function w is defined by

$$(-\Delta_y)^r w(y) = C\int_{\mathbb{R}^n} \frac{w(y) - w(z)}{|y-z|^{n+2r}} dz,\ \forall y \in \mathbb{R}^n. \tag{2.1}$$

We define the fractional-order Sobolev space by

$$H^r(\Gamma_1) = \Big\{w \in L^2(\Gamma_1) : \int_{\Gamma_1}\int_{\Gamma_1} \frac{|w(y) - w(z)|^2}{|y-z|^{n+2r}} dydz < \infty\Big\}, \tag{2.2}$$

with the norm

$$||w||_{H^r(\Gamma_1)} = \sqrt{\int_{\Gamma_1} |w|^2 dy + \int_{\Gamma_1}\int_{\Gamma_1} \frac{|w(y) - w(z)|^2}{|y-z|^{n+2r}} dydz}. \tag{2.3}$$

Set

$$H_0^r(\Gamma_1) = \{w \in H^r(\Gamma_1) : w = 0\ a.e.\ in\ \Gamma_2\}, \tag{2.4}$$

then $H_0^r(\Gamma_1)$ is a closed linear subspace of $H^r(\Gamma_1)$, where

$$||w||_{H_0^r(\Gamma_1)} = \sqrt{\int_{\Gamma_1}\int_{\Gamma_1} \frac{w(y) - w(z)|^2}{|y-z|^{n+2r}} dydz}. \tag{2.5}$$

The space $H_0^r(\Gamma_1)$ is a Hilbert space with

$$\langle w, v\rangle_{H_0^r(\Gamma_1)} = \int_{\Gamma_1}\int_{\Gamma_1} \frac{(w(y) - w(z))(v(y) - v(z))}{|y-z|^{n+2r}} dydz. \tag{2.6}$$

Let Γ_1 be a bounded domain, then

1) The embedding $H_0^r(\Gamma_1) \hookrightarrow L^p(\Gamma_1)$ is compact $\forall 1 \leq p < 2_r^\star$;
2) The embedding $H_0^r(\Gamma_1) \hookrightarrow L^{2_r^\star}(\Gamma_1)$ is continuous.

For any $1 \leq s \leq 2_r^\star$, $\exists C_0 = C_0(n, s, r > 0)$ such that $\forall v \in H_0^r(\Gamma_1)$

$$\|w\|_{L^s(\Gamma_1)} \leq C_0 \int_{\Gamma_1}\int_{\Gamma_1} \frac{v(y) - v(z)|^2}{|y - z|^{n+2r}} dydz. \tag{2.7}$$

For any $s \in [1, 2_r^*]$ and any bounded sequence $\{u_j\}_{j=1}^\infty$ in $H_0^r(\Gamma_1)$ there exists v in $L^s(\mathbb{R}^n)$, with $v = 0$ a.e. in $\mathbb{R}^n \setminus \Gamma_1$, such that up to a sub-sequence, still written as $\{v_j\}_{j=1}^\infty$,

$$v_j \to v \text{ strongly in } L^s(\Gamma_1) \text{ as } j \to \infty. \tag{2.8}$$

We have the following property for any h positive number,

$$|\log(z)| \leq \frac{1}{h} z^h, \forall z \in [1, +\infty). \tag{2.9}$$

Let $\theta \in (0, 1)$, and $p_\theta \in]p_0, p_1[\subset [1, +\infty[$ with $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have the following inequality,

$$\|w\|_{p_\theta} \leq \|w\|_{p_0}^{1-\theta}\|w\|_{p_1}^\theta, \quad \forall w \in L^{p_0}(\Gamma_1) \cap L^{p_1}(\Gamma_1).$$

3. Important theories and properties

We denote by $Z = H_0^r(\Gamma_1) \setminus \{0\}$. The associate energy $\mathcal{E}$ of (1.4) is defined by

$$\mathcal{E}(t) = \frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{K}(w), \tag{3.1}$$

with the functional $\mathcal{K} \in (Z, \mathbb{R})$ associated with problem (1.4) is given by

$$\mathcal{K}(w) = \frac{1}{2}\|w\|_Z^2 + \frac{1}{p^2}\|w\|_p^p - \frac{1}{p}\int_{\Gamma_1} |w|^p \log(|w|)dy, \tag{3.2}$$

the Nehari functional $\mathcal{J} \in (Z, \mathbb{R})$, defined by,

$$\mathcal{J}(w) =< \mathcal{K}'(w), w >= \|w\|_Z^2 - \int_{\Gamma_1} |w|^p \log(|w|)dy, \tag{3.3}$$

where $< ., . >$ denotes the dual pairing between $H_0^r(\Gamma_1)$ and $\left(H_0^r(\Gamma_1)\right)'$.

We define the following two groups:

$$\mathcal{W} = \{w \in Z : \mathcal{J}(w) > 0, \mathcal{K}(w) < d\}, \tag{3.4}$$

and

$$\mathcal{V} = \{w \in Z : \mathcal{J}(w) < 0, \mathcal{K}(w) < d\}, \tag{3.5}$$

and

$$\mathcal{N} = \{w \in Z : \mathcal{J}(w) = 0\}. \tag{3.6}$$

Now, define d as

$$d = \inf_{w \in Z} \{\sup_{\mu \geq 0} \mathcal{K}(\mu w)\}, \tag{3.7}$$

which characterized as

$$d = \inf_{w \in \mathcal{N}} \mathcal{K}(w). \tag{3.8}$$

For any α satisfying $p < p + \alpha \leq 2_r^\star$, we put

$$r(\alpha) = \left(\frac{\alpha}{k^{p+\alpha}}\right)^{\frac{1}{p+\alpha-2}},$$

where k is the optimal embedding constant of

$$H_0^r(\Gamma_1) \hookrightarrow L^{p+\alpha}(\Gamma_1).$$

i.e.,

$$k = \inf_{w \in H_0^r(\Gamma_1) \setminus \{0\}} \frac{\|w\|_{H_0^r(\Gamma_1)}}{\|w\|_{p+\alpha}}.$$

Lemma 3.1. *Let $w \in H_0^r(\Gamma_1) \setminus \{0\}$ and $p < p + \alpha \leq 2_r^\star, r \in (0, 1)$. We have*

1) *If $0 < \|w\|_{H_0^r(\Gamma_1)} \leq r(\alpha)$, then $\mathcal{J}(w) > 0$.*
2) *If $\mathcal{J}(w) \leq 0$, then $\|w\|_{H_0^r(\Gamma_1)} > r(\alpha), \alpha > 0$.*

Proof. Let $w \in Z$, according to (2.9), we have

$$\log(|w(y)|) < \frac{|w(y)|^\alpha}{\alpha}, \quad \forall y \in \Gamma_1.$$

From the definition of $\mathcal{J}$, we obtain

$$\begin{aligned}\mathcal{J}(w) &= \|w\|_{H_0^r(\Gamma_1)}^2 - \int_{\Gamma_1} |w|^p \log |w| dy \\ &> \|w\|_{H_0^r(\Gamma_1)}^2 - \frac{\|w\|_{p+\alpha}^{p+\alpha}}{\alpha},\end{aligned}$$

by the definition of k, we know

$$\|w\|_{p+\alpha}^{p+\alpha} \leq k^{p+\alpha} \|w\|_{H_0^r(\Gamma_1)}^{p+\alpha},$$

then

$$\begin{aligned}\mathcal{J}(w) &> \|w\|_{H_0^r(\Gamma_1)}^2 - \frac{k^{p+\alpha}}{\alpha} \|w\|_{H_0^r(\Gamma_1)}^{p+\alpha} \\ &= \|w\|_{H_0^r(\Gamma_1)}^2 \left(1 - \frac{k^{p+\alpha}}{\alpha} \|w\|_{H_0^r(\Gamma_1)}^{p+\alpha-2}\right).\end{aligned}$$

1) If $0 < \|w\|_{H_0^r(\Gamma_1)} \leq r(\alpha)$, then $\mathcal{J}(w) > 0$;
2) If $\mathcal{J}(w) \leq 0$, implies that $\|w\|_{H_0^r(\Gamma_1)} > r(\alpha)$.

We put

$$R(\alpha) = \left(\frac{\alpha}{K^{p+\alpha}}\right)^{\frac{1}{p+\alpha-2}} |\Gamma_1|^{\frac{\alpha}{p(p+\alpha-2)}},$$

and

$$K = \inf_{w \in Z} \frac{\|w\|_{H_0^r(\Gamma_1)}}{\|w\|_p},$$

and

$$\begin{cases} r^\star = \sup\limits_{\alpha \in (0, 2^\star - p]} R(\alpha), \\ r_\star = \sup\limits_{\alpha \in (0, 2^\star - p]} r(\alpha). \end{cases}$$

Lemma 3.2. *Let $\alpha \in (0, 2_r^\star - p]$. We have*

$$0 < r_\star \leq r^\star < \infty.$$

Proof.
- Since r and R are two continuous functions on a compact $[0, 2_r^\star - p]$, the $r^\star$ and $r_\star$ exists.
- Let $r(\alpha) \leq R(\alpha), \forall \alpha \in (0, 2_r^\star - p]$ and let $w \in H_0^r(\Gamma_1)$, then $w \in L^p(\Gamma_1) \cap L^{p+\alpha}(\Gamma_1)$, by Holder's inequality we have

$$\begin{aligned} \int_{\Gamma_1} |w|^p dy &\leq \left(\int_{\Gamma_1} dy\right)^{\frac{\alpha}{p+\alpha}} \left(\int_{\Gamma_1} |w|^{p+\alpha} dy\right)^{\frac{p}{P+\alpha}} \\ &= (|\Gamma_1|)^{\frac{\alpha}{p+\alpha}} \left(\int_{\Gamma_1} |w|^{p+\alpha} dy\right)^{\frac{p}{P+\alpha}}, \end{aligned}$$

then

$$\|w\|_p \leq (|\Gamma_1|)^{\frac{\alpha}{p(p+\alpha)}} \|w\|_{p+\alpha},$$

and

$$\begin{aligned} k &= \inf_{w \in Z} \frac{\|w\|_{H_0^r(\Gamma_1)}}{\|w\|_{p+\alpha}} \\ &\geq (|\Gamma_1|)^{\frac{-\alpha}{p(p+\alpha)}} \inf_{w \in Z} \frac{\|w\|_{H_0^r(\Gamma_1)}}{\|w\|_p} \\ &= (|\Gamma_1|)^{\frac{-\alpha}{p(p+\alpha)}} K, \end{aligned}$$

we obtain

$$
\begin{aligned}
r(\alpha) &= \left(\frac{\alpha}{k^{p+\alpha}}\right)^{\frac{1}{p+\alpha-2}} \\
&\leq \left(\frac{\alpha}{K^{p+\alpha}}|\Gamma_1|^{\frac{\alpha}{p}}\right)^{\frac{1}{p+\alpha-2}} \\
&\leq \left(\frac{\alpha}{K^{p+\alpha}}\right)^{\frac{1}{p+\alpha-2}} |\Gamma_1|^{\frac{\alpha}{p(p+\alpha-2)}} \\
&= R(\alpha),
\end{aligned}
$$

so

$$
r_\star = \sup_{\alpha\in(0,2_r^\star-p]} r(\alpha) \leq \sup_{\alpha\in(0,2_r^\star-p]} R(\alpha) = r^\star < \infty.
$$

Corollary 3.3. *Let $w \in H_0^r(\Gamma_1) \setminus \{0\}$ and $p < p+\alpha \leq 2_s^\star$. We have*

1) If $0 < \|w\|_{H_0^r(\Gamma_1)} \leq r_\star$, then $\mathcal{J}(w) > 0$.
2) If $\mathcal{J}(w) \leq 0$, then $\|w\|_{H_0^r(\Gamma_1)} > r_\star$.

Lemma 3.4. *Let $\alpha \in (0, 2_r^\star - p]$, we have*

$$
d \geq \frac{r_\star^2(p-2)}{2p}.
$$

Proof. Let $w \in \mathcal{N}$, we have $w \in H_0^r(\Gamma_1)\setminus\{0\}$ and $\mathcal{J}(w) = 0$, thus by Corollary 3.3, we obtain $\|w\|_{H_0^r(\Gamma_1)} \geq r_\star$, then

$$
\begin{aligned}
\mathcal{K}(w) &= \frac{1}{p}\mathcal{J}(w) + \frac{p-2}{2p}\|w\|_{H_0^r(\Gamma_1)}^2 + \frac{1}{p^2}\|w\|_p^p \\
&\geq \frac{p-2}{2p}r_\star^2.
\end{aligned}
$$

Let $w \in H_0^r(\Gamma_1)$, if $\mathcal{J}(w) < 0$, then there exists a $\mu^\star \in (0,1)$ such that

$$
\mathcal{J}(\mu^\star w) = 0.
$$

Let $w \in H_0^r(\Gamma_1)$, if $\mathcal{J}(w) < 0$, then

$$
\mathcal{J}(w) < p(\mathcal{K}(w) - d).
$$

Lemma 3.5. *We have*

1) $d = \inf_{w\in Z}\sup_{\mu>0}\mathcal{K}(\mu w)$ has a positive lower bound, namely

$$
d \geq \frac{R^p}{p^2}.
$$

2) There exists function $w \in \mathcal{N}$, such that $\mathcal{K}(w) = d$.

Proof. 1) Let $w \in H_0^r(\Gamma_1)$, we have

$$\sup_{\mu>0} \mathcal{K}(\mu w) = \mathcal{K}(\mu^* w) = \frac{1}{p^2}\|\mu^\star w\|_p^p, \tag{3.9}$$

by $\mu^\star w \in \mathcal{N}$, thus

$$\mathcal{K}(\mu^\star w) \geq d = \inf_{w\in\mathcal{N}} \mathcal{K}(w). \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$d = \inf_{w\in Z} \sup_{\mu>0} \mathcal{K}(\mu w) \geq d.$$

On the other hand, if $w \in \mathcal{N}$, we obtain the only critical point in $(0, +\infty)$ of the mapping $\mathcal{K}$ is $\mu^\star = 1$. Thus,

$$\sup_{\mu>0} \mathcal{K}(\mu w) = \mathcal{K}(w),$$

for $w \in \mathcal{N}$. Hence

$$\inf_{w\in H_0^r(\Gamma_1)} \sup_{\mu>0} \mathcal{K}(\mu w) \leq \inf_{w\in\mathcal{N}} \sup_{\mu>0} \mathcal{K}(\mu w) = \inf_{w\in\mathcal{N}} \mathcal{K}(w) = d,$$

we have $\mathcal{J}(\mu^\star w) = 0$. This implies

$$\|\mu^\star w\|_p \geq R,$$

yields

$$\sup_{\mu>0} \mathcal{K}(\mu w) \geq \frac{R^p}{p^2}.$$

2) Let $\{w_k\}_{k=1}^\infty \subset \mathcal{N}$ be minimizing sequence for $\mathcal{K}$ such that

$$\lim_{k\to+\infty} \mathcal{K}(w_k) = d.$$

On the other hand, we have $\{w_k\}_{k=1}^\infty$ is bounded in $H_0^r(\Gamma_1)$. Since $H_0^r(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ is compact, there exists a function w and a sub-sequence of $\{w_k\}_{k=1}^\infty$, still denoted by $\{w_k\}_{k=1}^\infty$, such that

$$\begin{aligned} w_k &\rightharpoonup w, \text{ in } H_0^r(\Gamma_1), \\ w_k &\rightharpoonup w, \text{ in } L^2(\Gamma_1), \\ w_k &\rightharpoonup w, \text{ in } \Gamma_1, \end{aligned}$$

we claim that

$$\lim_{k\to+\infty} \int_{\Gamma_1} |w_k| \log(|w_k|)dy = \int_{\Gamma_1} |w| \log(|w|)dy,$$

this implies

$$\lim_{k\to+\infty} |w_k|^p \log(|w_k|) = |w|^p \log(|w|), \text{ a.e }, y \in \Gamma_1,$$

and

$$\int_{\Gamma_1} \left||w|^{p-2}w\log|w|\right|^{\frac{p}{p-1}} dy = \int_{\{y\in\Gamma_1;|w|>1\}} ||w|^{p-2}w\log|w||^{\frac{p}{p-1}} dy + \int_{\{y\in\Gamma_1;|w|\le 1\}} ||w|^{p-2}w\log|w||^{\frac{p}{p-1}} dy,$$

we can obtain

$$\int_{\{y\in\Gamma_1;|w|\le 1\}} ||w|^{p-2}w\log|w||^{\frac{p}{p-1}} dy \le \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} |\Gamma_1|,\ \forall 0 \le t < +\infty.$$

We can choose now a constant $h > 0$ such that

$$\frac{p(p+h-1)}{p-1} \in [1, 2_r^\star].$$

Then we can infer that there must be a constant $C_\star > 0$ such that

$$||w||_{\frac{p(p+h-1)}{p-1}} \le C_\star ||w||_{H_0^r(\Gamma_1)},$$

by $\log(z) \le \frac{1}{h}z^h (h, z > 0)$, then

$$\begin{aligned}
&\int_{\{y\in\Gamma_1;|w|>1\}} ||w|^{p-2}w_n\log|w||^{\frac{p}{p-1}} dy \\
&\le h^{\frac{p}{p-1}} \int_{\{y\in\Gamma_1;|w|>1\}} |w|^{\frac{p(p+h-1)}{p-1}} dy \\
&\le h^{\frac{p}{p-1}} ||w||^{\frac{p(p+h-1)}{p-1}}_{\frac{p(p+h-1)}{p-1}} \\
&\le C_\star^{\frac{p(p+h-1)}{p-1}} h^{\frac{p}{p-1}} ||w||^{\frac{p(p+h-1)}{p-1}}_{H_0^r(\Gamma_1)} \\
&\le C_\star^{\frac{p(p+h-1)}{p-1}} h^{\frac{p}{p-1}} \left(\frac{2pd}{p-2}\right)^{\frac{p(p+h-1)}{2(p-1)}}.
\end{aligned}$$

Then, for any $t \in [0, +\infty)$, we have

$$\begin{aligned}
&\int_{\Gamma_1} \left||w|^{p-2}w\log|w|\right|^{\frac{p}{p-1}} dy \\
&\le C_\star^{\frac{q(p+h-1)}{p-1}} h^{\frac{p}{p-1}} \left(\frac{2pd}{p-2}\right)^{\frac{p(p+h-1)}{2(p-1)}} + \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} |\Gamma_1|.
\end{aligned}$$

We conclude that

$$\lim_{k\to+\infty} |w_k|^p \log(|w_k|) = |w|^p \log(|w|),\ \text{weakly in } L^{\frac{p}{p-1}}(\Gamma_1).$$

On the other hand, we have

$$
\begin{aligned}
&\left|\int_{\Gamma_1} |w_k|^p \log(|w_k|)dy - \int_{\Gamma_1} |w|^p \log(|w|)dy\right| \\
&\leq \left|\int_{\Gamma_1} (w_k - w)|w_k|^{p-2} w_k \log(|w_k|)dy\right| \\
&+ \left|\int_{\Gamma_1} w|w_k|^{p-2} w_k \log(|w_k|)dy - \int_{\Gamma_1} |w|^{p-2} w \log(|w|)dy\right| \\
&\leq C\|w_k - w\|_p + \left|\int_{\Gamma_1} w\left[|w_k|^{p-2} w_k \log(|w_k|) - |w|^{p-2} w \log(|w|)\right] dy\right|.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
\mathcal{K}(w) &= \frac{1}{2}\|w\|^2_{H_0^r(\Gamma_1)} + \frac{1}{p^2}\|w\|_p^p - \frac{1}{p}\int_{\Gamma_1} |w|^p \log(|w|)dy \\
&\leq \inf_{k\to+\infty}\left\{\frac{1}{2}\|w_k\|^2_{H_0^r(\Gamma_1)} + \frac{1}{p^2}\|w_k\|_p^p - \frac{1}{p}\int_{\Gamma_1} |w_k|^p \log(|w_k|)dy\right\} \\
&\leq \inf_{k\to+\infty}\{\mathcal{K}(w_k)\} = d.
\end{aligned}
$$

By $w_k \in \mathcal{N}$ we have $w_k \in H_0^r(\Gamma_1)$ and $\mathcal{J}(w_k) = 0$, then we obtain

$$\|w_k\|_p \geq R.$$

Hence

$$
\begin{aligned}
\mathcal{J}(w) &= \|w\|^2_{H_0^r(\Gamma_1)} - \int_{\Gamma_1} |w|^p \log(|w|)dy \\
&\leq \inf_{k\to+\infty}\left\{\|w_k\|^2_{H_0^r(\Gamma_1)} - \int_{\Gamma_1} |w_k|^p \log(|w_k|)dy\right\} \\
&\leq \inf_{k\to+\infty}\{\mathcal{J}(w_k)\} = 0.
\end{aligned}
$$

It remains to show that $\mathcal{J}(w) = 0$. Arguing by contradiction, if this is not true then we have $\mathcal{J}(w) < 0$, there exists a positive constant $\mu^\star$ such that $\mu^\star < 1$ and satisfying $\mathcal{J}(\mu^\star w) = 0$. Therefore, by definition of d, we obtain

$$
\begin{aligned}
0 < d \leq \mathcal{K}(\mu^\star w) &= \frac{1}{p^2}\|\mu^\star w\|_p^p \\
&\leq \frac{(\mu^\star)^p}{p^2}\lim_{k\to+\infty}\|w_k\|_p^p \\
&= (\mu^\star)^p \lim_{k\to+\infty}\mathcal{K}(w_k) = (\mu^\star)^p d < d,
\end{aligned}
$$

but this is a contradiction.

4. Global existence of solutions

Here, we state and prove the result regarding the global existence of solutions.

Definition 4.1. *A function*

$$w \in L^{\infty}(0,\infty, H_0^r(\Gamma_1)),$$

with

$$w_t \in L^{\infty}(0,\infty, L^p(\Gamma_1)),$$

is said to be a global (weak) solution of (1.4), if

$$w_0 \in L^{\infty}(0,\infty, H_0^r(\Gamma_1)),$$

$$w_1 \in L^{\infty}(0,\infty, L^2(\Gamma_1)),$$

and

$$\forall \phi \in L^{\infty}(0,\infty, H_0^r(\Gamma_1)),$$

$t \in \mathbb{R}_+^{\star}$

$$\int_{\Gamma_1}(w_t,\phi)dy + \int_0^t (w,\phi)_{H_0^r(\Gamma_1)}d\tau + + \int_0^t (w_t,\phi)_{H_0^r(\Gamma_1)}d\tau$$
$$= \int_{\Gamma_1}(w(y,t=0),\phi)dy + \int_0^t \Big(|w(y,\tau)|^{p-2}w(y,\tau)\log(|w(y,\tau)|),\phi(y,\tau)\Big)\,d\tau.$$

If a (weak) global solution $w \in C(0,\infty; H_0^r(\Gamma_1))$, it is said that w is a strong global solution of (1.4).

Theorem 4.2. *Let $w_0 \in H_0^r(\Gamma_1)$ and $w_1 \in L^2(\Gamma_1)$, suppose that $\mathcal{E}(t=0) < d$, and $\mathcal{J}(w_0) > 0$. Then problem (1.4) admits a global solution $w \in L^{\infty}(0,\infty, H_0^r(\Gamma_1))$, with $w_t \in L^{\infty}(0,\infty, L^2(\Gamma_1))$ and $w \in \mathcal{W}, \forall t \in \mathbb{R}_+^{\star}$.*

Proof. We will use the Galerkin method. For this end, we divide the proof into next steps

Step 1: By [21] there exists a sequence $(u_j)_j \subset C_0^{\infty}(\Gamma_1)$ of eigenfunctions of the fractional Laplace operator $(-\Delta_y)^r$, which is an orthonormal basis of $L^2(\Gamma_1)$ and an orthogonal basis of $H_0^r(\Gamma_1)$.
Let $\{V_m\}_{m\in\mathbb{N}}$ be the Galerkin space of the separable Banach space $H_0^r(\Gamma_1)$, i.e.,

$$V_n = Span\{u_1,u_2,\cdots,u_n\} \text{ and } \overline{\bigcup_{n\in\mathbb{N}} V_n} = H_0^r(\Gamma_1),$$

with $\left\{u_j\right\}_{j=1}^n$ is an orthonormal basis in $L^2(\Gamma_1)$.
Let $w_0 \in H_0^r(\Gamma_1)$, then we can find $w_{0n} \in V_n$. We shall find the approximate solutions of the following equality:

$$w_n = \sum_{j=1}^n g_j^n(t)u_j(x), \qquad j = 1,2,3,\ldots$$

satisfying

$$\begin{cases}(\partial_t^2 w(.,t),u_j)+(w(.,t),u_j)_{H_0^r(\Gamma_1)}+(\partial_t w(.,t),u_j)_{H_0^r(\Gamma_1)}\\ =(|w(.,t)|^{p-2}w(.,t)\log(|w(.,t)|),u_j),\ j=\overline{1,n}\\ w_n(.,t=0)=\sum_{j=1}^{n}A_j u_j\to w_0, n\to\infty \text{ in } W_0^{r,p}(\Gamma_1),\\ \partial_t w_n(.,t=0)=\sum_{j=1}^{m}B_j u_j\to w_1, n\to\infty \text{ in } L^2(\Gamma_1).\end{cases}$$

Substituting w_n into (1.4), we obtain

$$\begin{cases}g_j^{n''}+\mu_j g_j^n+\mu_j g_j^{n'}\\ =\sum_{l=1}^{m}\left(|g_j^l|^{p-2}g_j^l\log(|g_j^l|)\int_{\Gamma_1}u_l|u_l|^{p-2}u_j dy+|g_j^l|^{p-2}g_j^l\int_{\Gamma_1}u_l|u_l|^{p-2}\log(|u_l|)u_j dy\right)\\ g_j^n(t=0)=a_j,\quad j=1,\dots,n\\ g_j^{n'}(t=0)=b_j,\quad j=1,\dots,n.\end{cases} \tag{4.1}$$

Owing to well-known standard ODE theory, we can see that (4.1) drives to a system of ODEs in t that admits a local solution $0\le w_n(t), 0\le t\le T_n$.

Step 2: By multiplication of problem (1.4) by $g_j^{n'}$, summing for j, we obtain

$$\begin{aligned}&\int_{\Gamma_1}\partial_t^2 w_n(y,\tau)\partial_t w_n(y,\tau)dy\\ +&\int_{\Gamma_1}\int_{\Gamma_1}\frac{|w_n-w_n(z,t)|^{p-2}(w_n-w_n(z,t))(\partial_t w_n-\partial_t w_n(z,t))}{|y-z|^{n+rp}}dydz\\ +&\int_{\Gamma_1}w_n(y,\tau)|w_n(y,\tau)|^{p-2}\partial_t w_n(y,\tau)dy\\ =&\int_{\Gamma_1}w_n(y,\tau)|w_n(y,\tau)|^{p-2}\log\left(|w_n(y,\tau)|\right)\partial_t w_n(y,\tau)dy,\end{aligned}$$

integrating the above equation with respect to τ, we obtain $\forall t\in\mathbb{R}_+^\star$

$$\begin{aligned}&\tfrac{1}{2}\int_0^t\frac{d}{dt}\left(\int_{\Gamma_1}|\partial_t w_m(y,\tau)|^2dy\right)d\tau+\frac{1}{2}\int_0^t\frac{d}{dt}\left(\int_{\Gamma_1}\int_{\Gamma_1}\frac{(w_n(y,\tau)-w_n(z,\tau))^2}{|y-z|^{n+2r}}dydz\right)d\tau\\ &+\int_0^t\|\partial_t w_n(z,\tau)\|^2_{W_0^{r,2}(\Gamma_1)}d\tau=\frac{1}{p}\int_0^t\frac{d}{dt}\int_{\Gamma_1}w_n(y,\tau)|w_n(y,\tau)|^{p-2}\log\left(|w_n(y,\tau)|\right)\partial_t w_n(y,\tau)dyd\tau,\end{aligned}$$

$\forall t\in\mathbb{R}_+^\star$, we obtain

$$\begin{aligned}&\frac{1}{2}\|\partial_t w_n(.,t)\|_2^2-\frac{1}{2}\|\partial_t w_n(.,t=0)\|_2^2+\frac{1}{2}\|w_n(.,t)\|^2_{H_0^r(\Gamma_1)}-\frac{1}{2}\|w_n(.,t=0)\|^2_{H_0^r(\Gamma_1)}\\ +&\int_0^t\|\partial_t w_n(z,\tau)\|^2_{H_0^r(\Gamma_1)}d\tau\\ =&\frac{1}{p}\int_{\Gamma_1}|w_n(y,\tau)|^p\log\left(|w_n(y,\tau)|\right)dy\\ -&\frac{1}{p}\int_{\Gamma_1}|w_n(y,t=0)|^p\log\left(|w_n(y,t=0)|\right)dy+\frac{1}{p^2}\|w_n(.,t=0)\|_p^p-\frac{1}{p^2}\|w_n(.,t)\|_p^p,\end{aligned}$$

so

$$\mathcal{E}_n(t) \le \mathcal{E}_n(t=0), \quad t \in [0, T_n], \tag{4.2}$$

where

$$\mathcal{E}_n(t) = \frac{1}{2}\|\partial_t w_n\|_2^2 + \mathcal{K}(w_n).$$

For n large enough, we can obtain $\mathcal{E}_n(t=0) < d$ and then $\mathcal{E}(t=0) < d$.
Then by (4.2), we have

$$\mathcal{E}_n(t) = \frac{1}{2}\|\partial_t w_n\|_2^2 + \mathcal{K}(w_n) < d. \tag{4.3}$$

According to $w_0 \in \mathcal{W}$, we can find that $\partial_t w_m \in \mathcal{W}$. Next, for $t \in [0, T_n]$, we will prove that $w_n \in \mathcal{W}$. Indeed, if not the case, there exist $t_2 \in (0, T_n]$ such that $w_n(t_2) = 0$ and $\mathcal{J}(w_n(t_2)) = 0$, then $w(t_2) \in \mathcal{N}$. Then $\mathcal{K}(w_n(t_2)) \ge d = \inf_{w\in\mathcal{N}} \mathcal{K}(w)$, which contradicts (4.3). Then, for sufficiently large n and $\forall t \in [0, T_n]$, we have $w_n \in \mathcal{W}$.
By (4.3), then $w_n \in \mathcal{W}$ and

$$\mathcal{K}(w_n) = \frac{(p-2)}{2p}\|w_n\|^p_{H_0^r(\Gamma_1)} + \frac{1}{p^2}\|w_n\|_p^p + \frac{1}{p}\mathcal{J}(w_n).$$

Then for $t \in [0, T_n]$ and n large enough, we have

$$\frac{1}{2}\|w_{nt}\|_2^2 + \frac{(p-2)}{2p}\|w_n\|^p_{H_0^r(\Gamma_1)} + \frac{1}{p^2}\|w_n\|_p^p < d,$$

which gives, $\forall 0 \le t \le T_n$

$$\begin{cases} \|w_{nt}\|_2^2 < 2d, \\ \|w_n\|^p_{H_0^r(\Gamma_1)} \le \frac{2pd}{p-2}, \\ \|w_n\|_p^p < dp^2. \end{cases} \tag{4.4}$$

So $T_n = +\infty$. Then we know (4.4) and $w_n \in \mathcal{W}, \forall 0 \le t < +\infty$.
Then

$$\begin{aligned} &\int_{\Gamma_1} \left| |w_n|^{p-2} w_n \log |w_n| \right|^{\frac{p}{p-1}} dy \\ = &\int_{\{y\in\Gamma_1; |w_n|>1\}} ||w_n|^{p-2} w_n \log |w_n||^{\frac{p}{p-1}} dy \\ + &\int_{\{y\in\Gamma_1; |w_n|\le 1\}} ||w_n|^{p-2} w_n \log |w_n||^{\frac{p}{p-1}} dy, \end{aligned}$$

we can get

$$\int_{\{y\in\Gamma_1; |w_n|\le 1\}} ||w_n|^{p-2} w_n \log |w_n||^{\frac{p}{p-1}} dy \le \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} |\Gamma_1|, \ \forall 0 \le t < +\infty.$$

We can choose now a constant $h > 0$ such that

$$\frac{p(p+h-1)}{p-1} \in [1, 2_r^\star].$$

Then we can infer that there must be a constant $C_\star > 0$ such that

$$\|w\|_{\frac{p(p+h-1)}{p-1}} \le C_\star \|w\|_{H_0^r(\Gamma_1)},$$

by $\log(z) \le \frac{1}{h} z^h$, $(h, z > 0)$, then

$$\begin{aligned}
&\int_{\{y\in\Gamma_1;|w_n|>1\}} \||w_n|^{p-2} w_n \log|w_n|\|^{\frac{p}{p-1}} dy \\
&\le h^{\frac{p}{p-1}} \int_{\{y\in\Gamma_1;|w_n|>1\}} |w_n|^{\frac{p(p+h-1)}{p-1}} dy \\
&\le h^{\frac{p}{p-1}} \|w_n\|_{\frac{p(p+h-1)}{p-1}}^{\frac{p(p+h-1)}{p-1}} \\
&\le C_\star^{\frac{p(p+h-1)}{p-1}} h^{\frac{p}{p-1}} \|w\|_{H_0^r(\Gamma_1)}^{\frac{p(p+h-1)}{p-1}} \\
&\le C_\star^{\frac{p(p+h-1)}{p-1}} h^{\frac{p}{p-1}} \left(\frac{2pd}{p-2}\right)^{\frac{p(p+h-1)}{2(p-1)}}.
\end{aligned}$$

Then, for sufficiently large n and for any $0 \le t < +\infty$, we have

$$\begin{aligned}
&\int_{\Gamma_1} \left||w_n|^{p-2} w_n \log|w_n|\right|^{\frac{p}{p-1}} dy \\
&\le C_\star^{\frac{q(p+h-1)}{p-1}} h^{\frac{p}{p-1}} \left(\frac{2pd}{p-2}\right)^{\frac{p(p+h-1)}{2(p-1)}} + \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} |\Gamma_1|.
\end{aligned}$$

Step 3: We see that there must be a function $w = w \in L^\infty(0, \infty, H_0^r(\Gamma_1))$ with $\partial_t w \in L^\infty(0, \infty, L^2(\Gamma_1))$, $\xi \in L^2(0, \infty, L^{\frac{q}{q-1}}(\Gamma_1))$ and a subsequence of $\{w_i\}_{i=1}^n$, as $n \to \infty$, such that,

$$\begin{aligned}
&w_n \rightharpoonup^\star w \text{ in } L^\infty(0, \infty, H_0^r(\Gamma_1)) \text{ and } w_n \rightharpoonup w \text{ in } L^2(0, \infty, H_0^r(\Gamma_1)) \\
&\text{ and } w_n \to w \text{ in } \Gamma \times \mathbb{R}_+^\star \\
&\partial_t w_n \rightharpoonup^\star \partial_t w \text{ in } L^\infty(0, \infty, L^2(\Gamma_1)) \text{ and } \partial_t w_n \rightharpoonup \partial_t w \text{ in } L^2(0, \infty, L^2(\Gamma_1)) \\
&\text{ and } \partial_t w_n \to \partial_t w \text{ in } \Gamma \times \mathbb{R}_+^\star \\
&|w_n|^{q-2} w_n \log(|w_n|) \to \xi \text{ in } L^\infty(0, \infty, L^{\frac{q}{q-1}}(\Gamma_1)) \\
&\text{ and } |w_n|^{q-2} w_n \log(|w_n|) \to \xi \text{ in } L^2(0, \infty, L^{\frac{q}{q-1}}(\Gamma_1)).
\end{aligned}$$

As in [9], the injection

$$\{w : \ w \in L^2(0, \infty, H_0^r(\Gamma_1)), \partial_t w \in L^2(0, \infty, L^2(\Gamma_1))\} \hookrightarrow L^2(0, \infty, L^p(\Gamma_1)),$$

is compact. We know

$$w_n \to w, \text{ strongly in } L^2(0,\infty, L^p(\Gamma_1)).$$

Then it follows from the convergence of w_n and w_{nt} that

$$\begin{aligned}
&\frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\|w\|_{H_0^r(\Gamma_1)}^2 + \frac{1}{p^2}\|w\|_p^p \\
&\leq \inf_{n\to+\infty}\left\{\frac{1}{2}\|\partial_t w_n\|_2^2 + \frac{1}{2}\|w_n\|_{H_0^r(\Gamma_1)}^2 + \frac{1}{p}\|w_n\|_p^p + \frac{1}{p^2}\|w_n\|_p^p\right\} \\
&\leq \inf_{n\to+\infty}\left\{\frac{1}{2}\|\partial_t w_n\|_2^2 + \mathcal{K}(w_n) + \frac{1}{p}\int_{\Gamma_1}|w_n|^p\log(|w_n|)dy\right\} \\
&\leq \inf_{n\to+\infty}\left\{\mathcal{E}_n(t) + \frac{1}{p}\int_{\Gamma_1}|w_n|^p\log(|w_n|)dy\right\} \\
&\leq \inf_{n\to+\infty}\left\{\mathcal{E}_n(t=0) + \frac{1}{p}\int_{\Gamma_1}|w_n|^p\log(|w_n|)dy\right\} \\
&\leq \mathcal{E}(t=0) + \frac{1}{p}\int_{\Gamma_1}|w|^p\log(|w|)dy,
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}\|w_t\|_2^2 + \mathcal{K}(w) \\
&= \frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\|w\|_{H_0^r(\Gamma_1)}^2 + \frac{1}{p}\|w\|_p^p + \frac{1}{p^2}\|w\|_p^p - \frac{1}{p}\int_{\Gamma_1}|w|^p\log(|w|)dy \\
&\leq \mathcal{E}(t=0).
\end{aligned}$$

Therefore, by $\mathcal{E}(t) \leq \mathcal{E}(t=0)$ for a.e. $0 \leq t < +\infty$ and $w_0 \in \mathcal{W}$, it is easy to prove $w \in \mathcal{W}$ for $0 \leq t < +\infty$.

Lemma 4.3. *Let w be a weak solution of (1.4), where T is the maximum existence time. Thus*

1) If $\mathcal{E}(t=0) < d$, $w_0 \in \mathcal{W}$, then $w \in \mathcal{W}$, for $0 \leq t < T$;
2) If $\mathcal{E}(t=0) \geq d$, $(w_0, w_1) \geq 0$, then $w \in \mathcal{V}$, and for $0 \leq t < T$, supply that $w_0 \in \mathcal{V}$.

Proof. Let T be the maximal existence time of the weak solution of w. We have

$$\frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{K}(w) \leq \frac{1}{2}\|w_1\|_2^2 + \mathcal{K}(w_0) < d, \quad \forall 0 \leq t < T. \tag{4.5}$$

1) Case of $\mathcal{E}(t=0) < d$. By contradiction, if not the case, then for $0 < t_0 < T$ we have $\mathcal{J}(w) < 0, \forall 0 \leq t < t_0$ and $\mathcal{J}(t_0) = 0$.
By claim 2) of Corollary 3.3, we have $\|w\|_{H_0^r(\Gamma_1)} \geq r^\star > 0$, for $0 \leq t < t_0$, then $w(t_0) \neq 0$. Thus $w(t_0) \in \mathcal{N}$ and $\mathcal{K}(w(t_0)) \geq 0$, which contradicts $\mathcal{K}(w(t_0)) \leq \mathcal{E}(t_0) < \mathcal{E}(t=0) < d$.

2) Case of $\mathcal{E}(t=0) \geq d$ and $(w_0, w_1) \geq 0$. By contradiction, if not the case, then there exists surely a $0 < t_0 < T$ such that $\mathcal{J}(w) < 0$, for $0 \leq t < t_1$ and $\mathcal{J}(t_1) = 0$.
By Corollary 3.3, the claim 2), we have $\|w\|_{H_0^r(\Gamma_1)} \geq r^\star > 0$, for $0 \leq t < t_0$, then $w(t_1) \neq 0$. Thus $w(t_0) \in \mathcal{N}$ and $\mathcal{K}(w(t_0)) \geq 0$. By definition of $\mathcal{E}(t)$, we have

$$\mathcal{E}(t_1) = \frac{1}{2}\|w(t_1)\|_2^2 + \mathcal{K}(w(t_1)) \leq \mathcal{E}(t=0) < d,$$

then $\mathcal{K}(w(t_1)) \leq d$ and $\|w(t_1)\|_2^2 = 0$. We first introduce an auxiliary function

$$\mathcal{M}(t) = \|w\|^2,$$

and

$$\mathcal{M}'(t) = (\partial_t w, w) + (w, \partial_t w) = 2(\partial_t w, w),$$

and

$$\mathcal{M}''(t) = 2(\partial_t w, w) + 2\|\partial_t w\|^2, \tag{4.6}$$

by (4.6), we have,

$$\mathcal{M}''(t) \geq 2\|\partial_t w\|^2 - 2\mathcal{J}(w). \tag{4.7}$$

Hence, we obtain

$$\mathcal{M}'(0) = 2(\partial_1 w, w_0) \geq 0,$$

then $\mathcal{M}'(t)$ is strictly increasing for $\leq t < t_1$. As $\mathcal{M}'(0) \geq 0$, we have $\mathcal{M}'(t) = 2(\partial_t w, w) \geq \mathcal{M}'(0) > 0$, which conflicts with $\|w(t_1)\|_2^2 = 0$.

Lemma 4.4. *Let* $(w_0, w_1) \in H_0^r(\Gamma_1) \times L^2(\Gamma_1)$*, suppose that* $0 < \mathcal{E}(t=0) < d$*. Then*

1) If $w_0 \in \mathcal{W}$ *, then* $w \in \mathcal{W}$*, for* $0 \leq t < T$*,*
2) If $w_0 \in \mathcal{V}$*, then* $w \in \mathcal{V}$*, for* $0 \leq t < T$*.*

Proof. Let T be the maximal existence time of the weak solution of w. We have

$$\frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{K}(w) \leq \frac{1}{2}\|w_1\|_2^2 + \mathcal{K}(w_0) < d, \quad \forall t \in [0, T). \tag{4.8}$$

1) We claim that $w \in \mathcal{W}, \forall 0 \leq t < T$. By contradiction, if not, then there must exist a $t_0 \in (0, T)$ such that $w(t_0) \in \partial\mathcal{W}$, and then we have $\mathcal{J}(w) = 0$. $\mathcal{K}(w_0) \geq d$ contradicts the assumption (4.8).

5. Conclusions and challenges

The present article is concerned with the wave equation involving the fractional Laplacian with logarithmic nonlinearity. With the aid of techniques from variational methods, we proved the global existence of weak solutions by the Galerkin approximation argument.

5.1. *Different formulas for fractional Laplacian operator*

It is more difficult situation in the case where the fractional Laplacian is nonlinear. We assume that the space variable $y \in \mathbb{R}^n$ and the fractional exponent are $0 < s < 1$. The next version is equivalent.

1) The first pseudo-differential operator given by the Fourier transform

$$(-\hat{\Delta}_y)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi).$$

2) Singular integral operator

$$(-\Delta_y)^s u(\xi) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(\xi) - u(y)}{|\xi - y|^{n+2s}} dy,$$

with this definition, it is the inverse of the Riesz integral operator $(-\Delta_y)^s u$. This one has a kernel $C_1 |\xi - y|^{n+2s}$, which is not integrable.

3) The β-harmonic extension: find the solution of the $(n + 1)$ problem

$$\nabla_y (y^{1-\beta} \nabla_y v) = 0, y \in \mathbb{R}^n, y \in \mathbb{R}_+; v(y, t = 0) = u(y).$$

If we put $\beta = 2s$, we obtain

$$(-\Delta_y)^s u(x) = -C_\beta \lim_{t \to 0} y^{1-\beta} \frac{\partial v}{\partial y},$$

when $s = 1/2$, *i.e.*, $(\beta = 1)$, the extended function v is harmonic and the operator is the Dirichlet-Neumann map on the space $\mathbb{R}^n$.

These three alternatives can be studied in probability and PDEs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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