People's Democratic Republic of Algeria

University of Ain Temouchent –Belhadj Bouchaib

Faculty of Science and Technology

Mechanical Engineering Department

Educational course

Title of Course

Mechanical Strength of Materials 2
(with Industrial Application Examples)

Course destined to students of:

3rd year License in Mechanical Construction

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Preface

The actual course presented in this document concerns the Mechanical Strength of Materials 2 (**Résistance des Matériaux 2**) which is a continuation of the first course of the Mechanical Strength of Materials 1 titled (**Résistance des Matériaux 1**) previously seen by students in their 2nd year of Mechanical Construction License. This course fully respects the entire program given in French language and cited in the canvas of the 3rd year of Mechanical Construction License.

We will approach in this course initially a reminder on pure bending of symmetrical beams, then we interest to the various methods of calculating the deflection of beams, energetic methods, combined loadings analysis, and the study of hyperstatic structures. This constitutes the main objective of our course.

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List of symbols

A: Beam cross-sectional area
b: Width of the rectangular cross-section of the beam
C: Torque
CG: Centroid or the gravity center
Cs: Safety coefficient
d: Diameter of the circular cross-section of the beam
D: Diameter
E: Modulus of elasticity or Young's modulus
F: Force
f: Deflection
$f_{\max}$: Maximum deflection
Fs: Safety factor
h: Thickness or height of the rectangular cross-section of the beam
I_z: Inertia moment
I_p: Polar inertia moment
L: Length of the beam
$Mf(x)$: Bending moment function of x
Mf_{max}: Maximal bending moment
M: Moment
M_t: Torsion moment
P: Weight
$q(x)$: Uniformly or non uniformly distributed load
R: Radius of the circular cross-section of the beam
R_x: Reaction according to the X axis
R_y: Reaction according to the Y axis
r: Radius of curvature
T: Torsion torque
$T(x)$: Shear force function of x
U: Elastic strain energy
u: Elastic strain-energy density
u_x: Deformation; displacement
u_y: Deflection
V: Volume
W: Work; Weight
$y(x)$: The equation of the curvature of the beam
y_{min}: The minimal deflection of the beam

γ: Shearing strain or angular strain
ε: Normal strain
θ: Rotation angle; slope; twist angle
ν: Poisson's ratio
μ: Shear modulus
σ: Normal stress
σ_{adm}: Allowable or admissible stress
σ_U: Ultimate strength
σ_Y: Yield stress
τ: Shearing stress

General Introduction

General introduction

This course constitutes a solid support for 3rd year License students in Mechanical Engineering branch, specialty Mechanical Construction, especially in terms of calculating the strength of materials, particularly with regard to the calculation of the mechanical resistance and the sizing of static or hyperstatic beams subjected to pure, unsymmetrical or combined bending.

In the **Chapter I**, we have presented a reminder concerning the calculation of the pure beam bending. Calculation of inertia moment, trenchant force or shear force and bending moment along the beam are presented in detail in this course. In addition, bending moment variations for beams subjected to uniformly and non uniformly distributed load are given in this document in order to calculate the tensile stresses in the beam, and then extract the maximal value of this stress. This maximum stress is compared to the yield stress of the beam material to see if this beam will be fractured or not. Finally, the calculation with demonstration of the tangential or the shear stress which is due to the shear force is show in this document.

In the **Chapter II**, different methods of the calculation of the beam deflection have been shown, we can cite for example: double-integration method, superposition method and moment area method. Several examples have been presented to better understand each method separately.

Chapter III is interested in the study and calculation of the elastic strain energy of different structures subjected to traction, compression, shearing, bending and torsion loadings. The knowledge of the elastic strain energy or the elastic strain-energy density and the strain energy verification criterion has two advantages; the first advantage is that we can calculate exactly the necessary strain energy to deform elastically a part and the second advantage is that this knowledge allows us to know whether or not our part will undergo plastic deformation after it absorbs impact energy.
A material that has a very high resilience modulus is more resistant to impact and does not deform plastically, also it absorbs and stores more elastic energy. On the other hand, a material that has a very high toughness modulus will need very high energy to make it break
Using the Castigliano theorem based on calculation of the derivative of the total elastic strain energy, we can calculate the displacement, rotation or the deflection at a given point of a bar, shaft or beam.

Chapter IV concerns the study of the unsymmetrical beams or the beams subjected to an unsymmetrical load, in other words, the beams which are not solicited in their planes of symmetry. Also, beams or shafts can be subjected simultaneously to many combined loadings as traction with bending or torsion with bending. In these cases, the use of the **Von Mises** or **Tresca** criterions is necessary to calculate the equivalent generated stresses in the structure and to verify if this latter can withstand the applied loading or not. Moreover, these criterions allow us to size the beam or the shaft and to calculate the minimum diameter with which the shaft can not deform plastically or breaks.

Finally, in the **Chapter V**, we have presented the different methods used to solve the hyperstatic system. Hyperstatic system is defined as a system in which its static equilibrium equations are unable to find the generated internal forces and their reactions in the structure. The hypestatic system is recognized when the number of the unknown actions of the supports is higher than the number of the static equilibrium equations of this system.

Also, the end of each chapter is completed by a directed works (DW) in order to allow students to apply the theoretical knowledge acquired in the course in the form of exercises.
In order to give to our students a good learning, and to make them interested to the real problems that will encounter in the industry and put them face to face with these problems, several examples of industrial applications of SOM (Strength of Materials) were presented to them in this document.
Finally, this course will provide to our students the basic notions and useful fundamental knowledge and information allowing them to choose, calculate and size the beams as best as possible. In addition, this module will help students acquire good study skills and competency that they will need in the world of work. Also, it aims to give them the fundamental knowledge of engineering principles, plus strong practical, theoretical, and transferable skills.

Chapter I

Pure bending of symmetrical beams

Chapter I: Pure bending of symmetrical beams

1. Introduction

After that the students have seen in their 2nd year of License the basic notions of calculation in resistance of materials of structures solicited by different loadings as tensile, compression, buckling, shearing, bending, etc; we remind them in this chapter, the principle of calculating of the mechanical resistance and sizing of beams in pure and plane bending.

2. A reminder of the different types of supports

2.1. Pinned support or fixed joint

Figure I-1 gives a schematization of the pinned support and its reactions. The reaction R_{Ax} along the x axis and the reaction R_{Ay} following the y axis are not zeros.

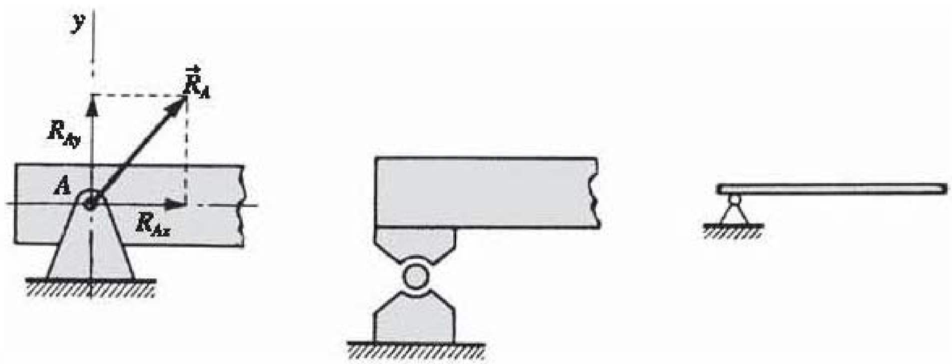


Figure I- 1: Pinned support or fixed joint [17]

2.2. Roller and simple supports or mobile joint

Figure I-2 gives a schematic diagram of the roller support and the simple support, and their reactions. The reaction R_{Ax} along the x axis is zero because the support is free to move along the x axis; while the reaction R_{Ay} following the y axis is not zero and this for both connections.

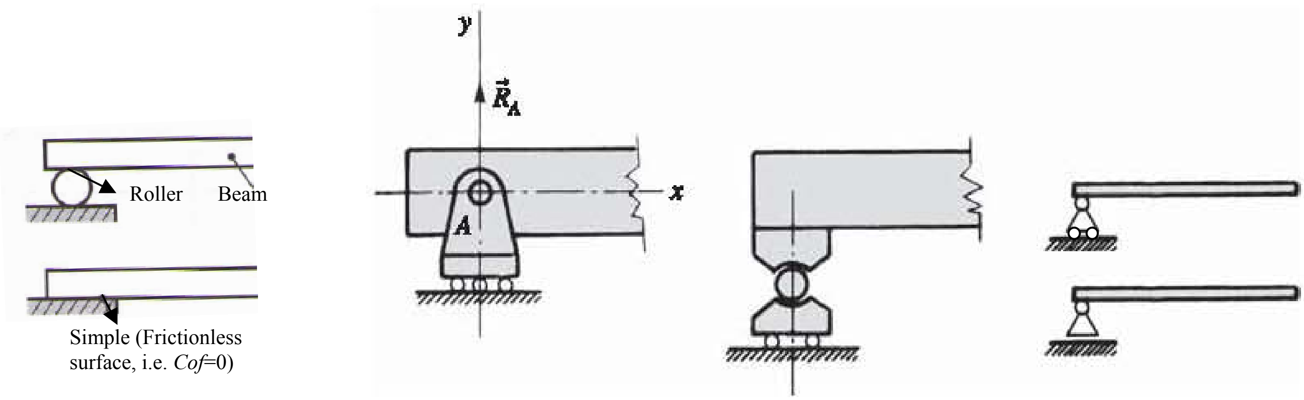


Figure I- 2: Roller support or mobile joint [2] & [17]

2.3. Fixed support or embedding

Figure I-3 gives a schematization of the fixed support or the embedding and its reactions and moment. The reaction R_{Ax} along the x axis and the reaction R_{Ay} following the y axis are not zeros. Also, the embedding moment M_A is not zero either.

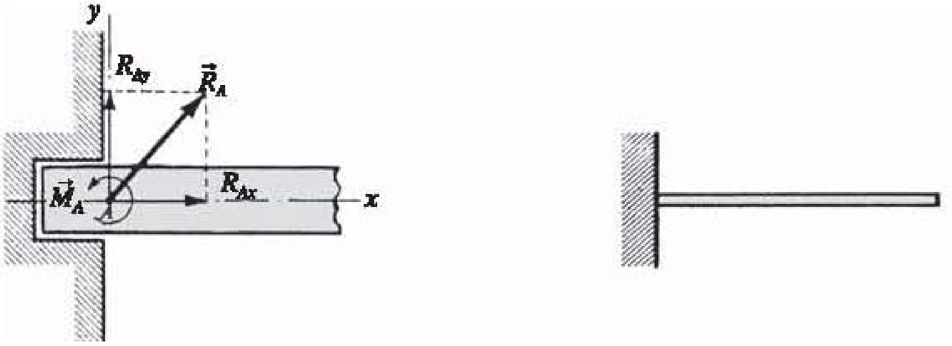


Figure I- 3: Fixed support or embedding [17]

3. A reminder on the calculation of the inertia moment of beam cross section

The following figure shows some types of beam cross sections. Each section has characteristics and a specific use for it, for example solid cross sections are more rigid than hollow cross sections; on the other hand, the latter are not heavy.

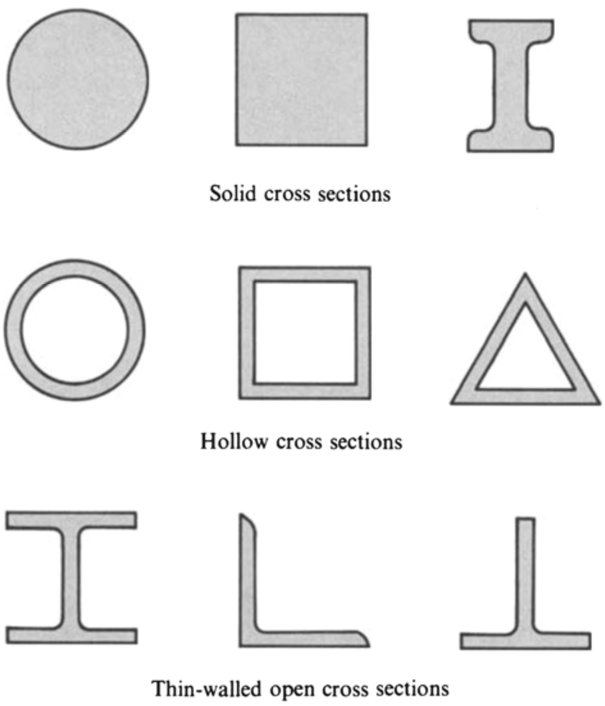


Figure I- 4: Typical cross sections of beams [3]

3.1. Calculation of the inertia moment of the beam cross-sectional area

Figure I-5 presents the beam axes and its cross sectional area.

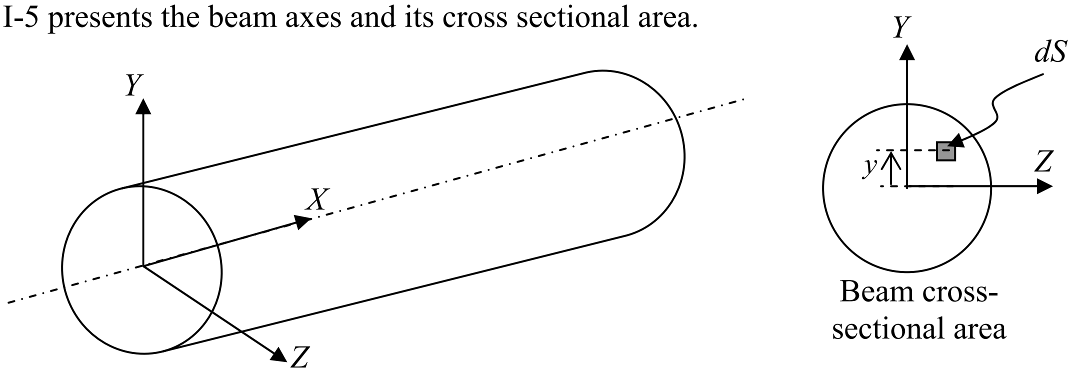


Figure I- 5: Beam with its symmetrical axes *Y* & Z and neutral axis *X*

The inertia moment of the cross plane beam section with respect to the Z axis is defined by the following integral:

$$I_z = \int y^2 dS \qquad \text{(I-1)}$$

I_z is the inertia moment of the cross-sectional area with respect to the *Z* axis and also with respect to the center of gravity of the cross section of the beam. If a beam made with a linearly elastic material is subjected to pure bending, then, the *Y* and *Z* axes are principal centroidal axes. Therefore, I_z will be equal to I_y. The polar moment of inertia is equal to:

$$I_p = I_z + I_y \qquad \text{(I-2)}$$

Some moments of inertia of some beam cross-sectional areas are indicated in the following table:

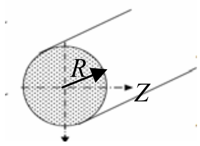
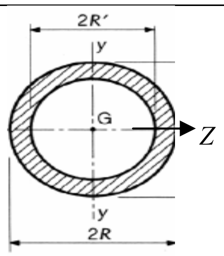
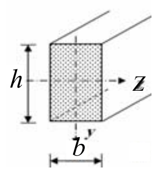
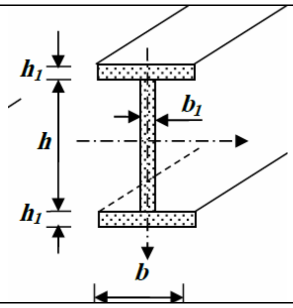
Cross-sectional areas	Moments of inertia ***Iz***
	$I_Z = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$ with $D = 2R$
	$I_Z = \frac{\pi}{4}\left(R^4 - R'^4\right)$
	$I_Z = \frac{bh^3}{12}$
	$I_Z = \frac{b*(h+2h_1)^3}{12} - \frac{(b-b_1)h^3}{12}$

Table I- 1: Inertia moments for some beam cross-sectional areas

4. Industrial examples to calculate shearing forces and bending moments

4.1. Example of a beam bending calculation of an hoisting drum

The example presented in the following figure concerns a hoisting drum; it is composed with a drum mounted on a cantilever beam; the drum can lift a heavy load by a cable wrapped on it. Also, an equivalent schematic (beam-support-load) of the hoisting drum system is presented in the following figure.

In dynamic, the following system must be sizing using the vibration and fatigue laws, but in static, the strength of materials laws should be used and this is exactly what will be presented in this chapter.

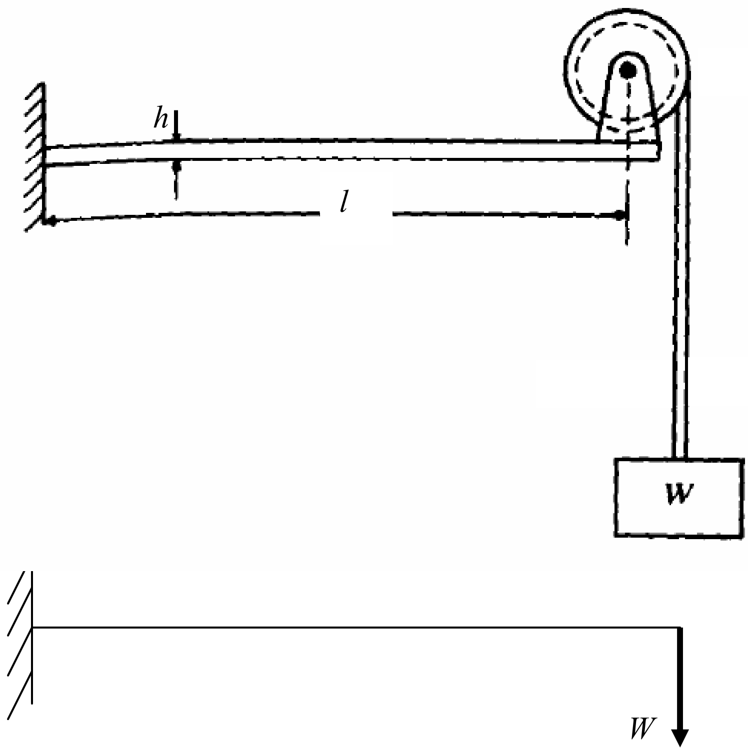


Figure I- 6: Cantilever beam of the hoisting drum [4]

In the above figure, W presents the weight of the load to be lifted; l is the length of the cantilever beam. The cross-sectional of the beam is rectangular having a width equal to b; the thickness of this cantilever beam is h.

To dimension this beam, we must search initially the maximum bending moment, then we calculate the maximum bending stress and we compare it with the material yield stress of this beam (the detail will be seen in section 7 of this chapter). To determine the maximum bending moment or the variation of this bending moment along the whole beam, we must already make at the beginning the forces and moments equilibrium to calculate the unknown reaction and embedding moment.

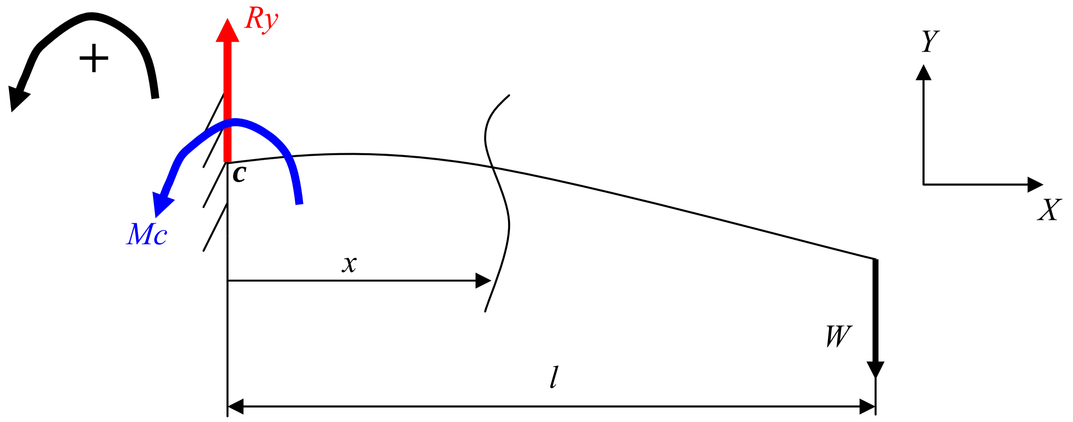


Figure I- 7: Determination of the embedding reaction and moment

Using the above figure (Figure I-7), we can determine the following static equilibrium equations and deduce later the unknown embedding reaction *Ry* and embedding moment *Mc*:

$$\sum F_{/Y} = 0 \Rightarrow Ry - W = 0 \Rightarrow Ry = W = mg$$

m is the mass of the load, g is the gravity;

$$\sum M_{/c} = 0 \Rightarrow Mc - Wl = 0 \Rightarrow Mc = Wl$$

4.1.1. Calculation of the shearing forces *T(x)*

a) Rules to determine the shear forces or the trenchant forces *T(x)*

We call the shear force (*T*) the internal transverse force and the bending moment (*Mf*) the internal moment. We try to determine these two parameters but at any point located in the longitudinal axis of the beam i.e. as a function of x. As a general rule when we cut the beam in two parts at x (Figure I-7), the shear force will be equal to the sum of the forces which are found in the section (*0-x*), the forces that are headed toward the upward are positive and the forces directing downward are negative. For our example, we obtain a constant $T(x)$ equal to:

$$0 < x < l \Rightarrow T(x) = Ry = T_{\max}$$

If I plot the shear force *T(x)* as a function of x, I then obtain the following curve:

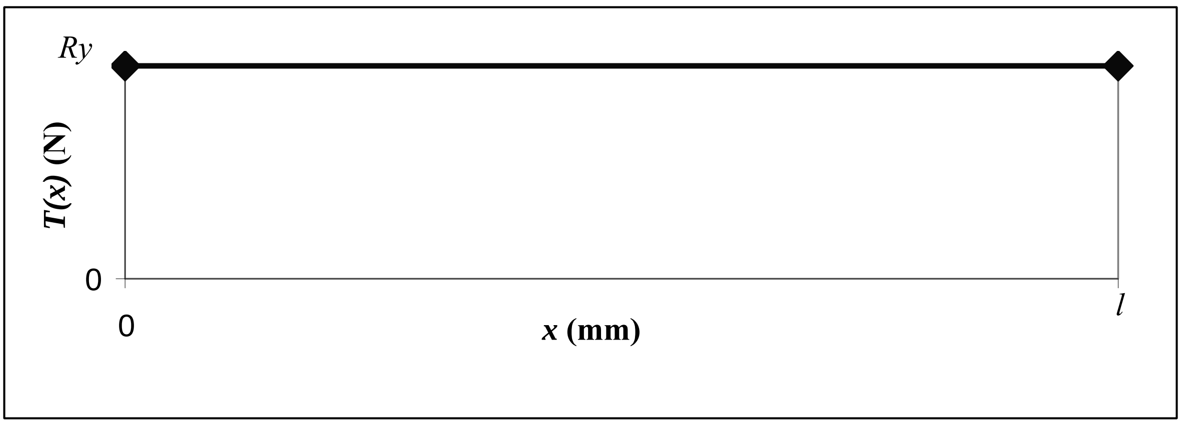


Figure I- 8: Variation of the shear force *T(x)* along the beam

4.1.2. Calculation of the bending moment *Mf(x)*

a) Rules to determine the bending moment *Mf(x)*

As a general rule, when we cut the beam in two parts at *x*, the bending moment will be equal to the sum of the moments and the forces moments existing in the part (0-x) with respect to *x*. The positive direction that must be followed to calculate the bending moments is that shown in black arrows in the figure below:

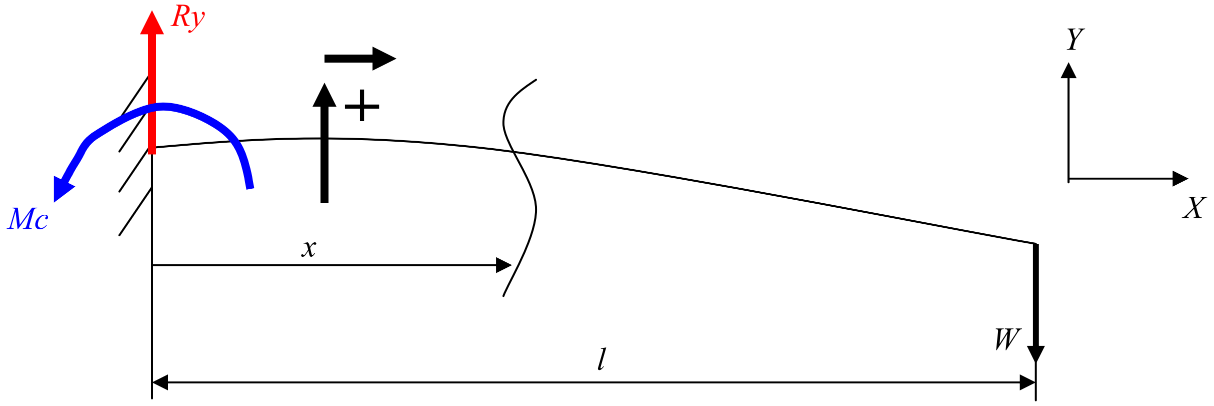


Figure I- 9: Determination method of the bending moment for the cantilever beam

For the previous example, we have:

$$0 < x < l \Rightarrow Mf(x) = Ry \times x - Mc = Wx - Wl = W(x - l)$$

If I plot the bending moment *Mf(x)* as a function of x, I then obtain:

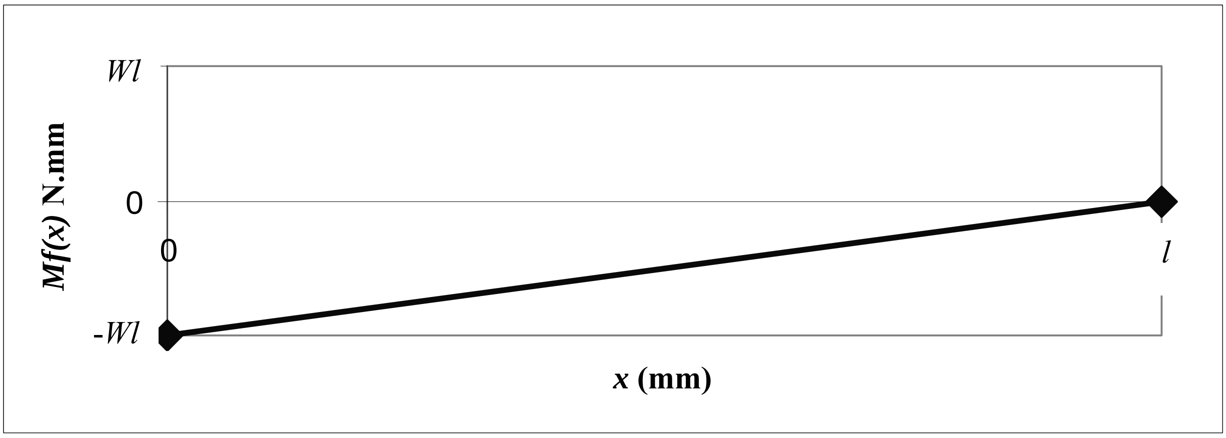
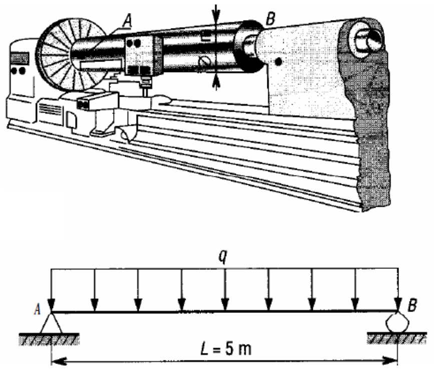


Figure I- 10: Variation of the bending moment *Mf(x)* along the beam

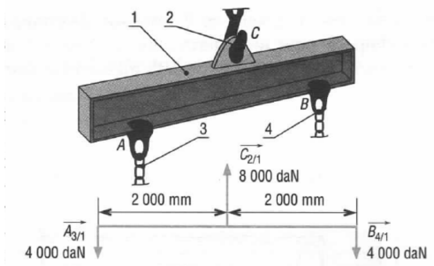
The maximum bending moment Mf_{max} is found in the embedding zone, it is equal to *Wl*, this tells us that the embedding zone is the most dangerous zone, i.e. the zone in which the beam would risk breaking in the first place.

4.2. Other industrial examples of beams with their equivalent (beam-support-load)

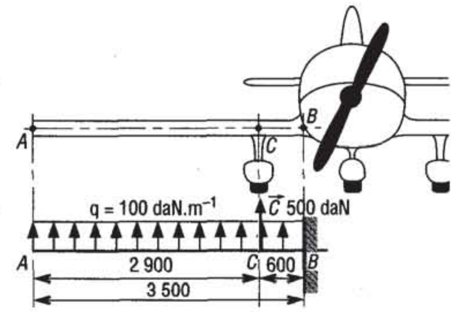
Note to that we apply the same manner used in the latter example to determine shear effort and bending moment.



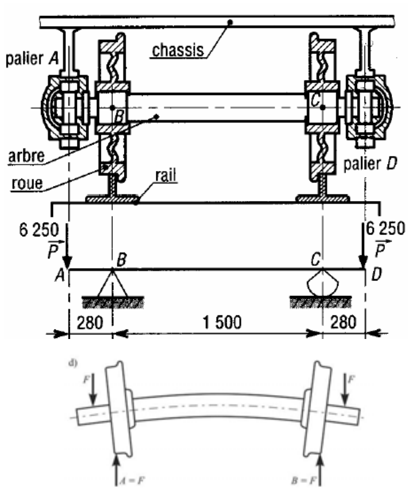
Beam mounted on lathe machine



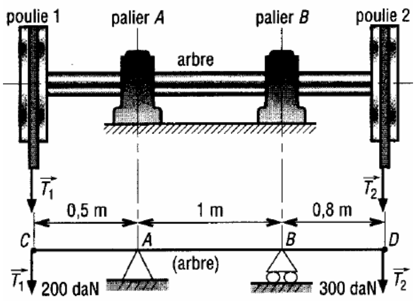
Lifting beam



Airplane wing embedded in the cabin



Wagon axle shaft



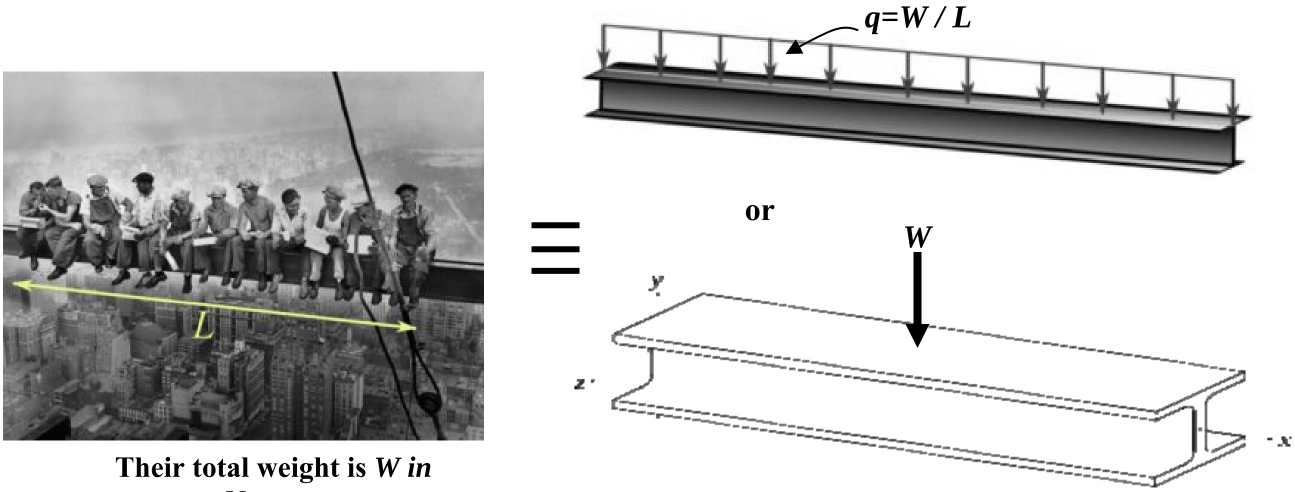
Shaft-pulley transmission system

Figure I- 11: Industrial examples of beams solicited in bending [2] & [5]

5. Uniformly distributed load

A beam loaded with a uniformly distributed load means that it supports a constant linear load along its length (i.e. a constant load divided by the length of the beam).

To better understand, here is an example illustrating the previous definition in the figure below. The weight of these different men who sit on the I-shaped beam can be modeled by a linear distributed load applied to the entire beam or by a point load applied to the centroid of the beam.

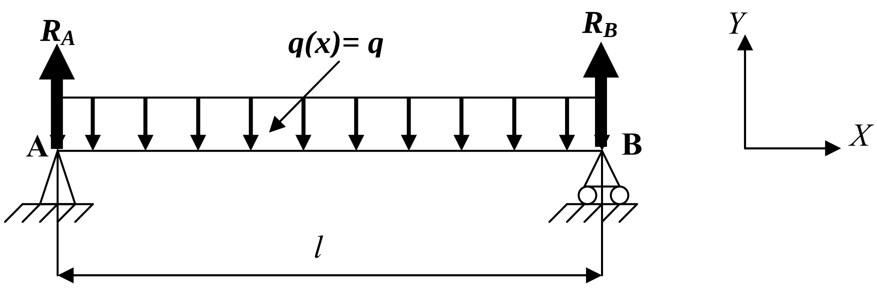


Their total weight is *W in* Newton

Figure I- 12: Illustrative example of uniformly distributed load *q* [18]

We show in the following examples of *T(x)* and *Mf(x)* calculation for a beam solicited by a constant and uniformly distributed load *q* (N/m):

Example 1:



Static equilibrium equations:

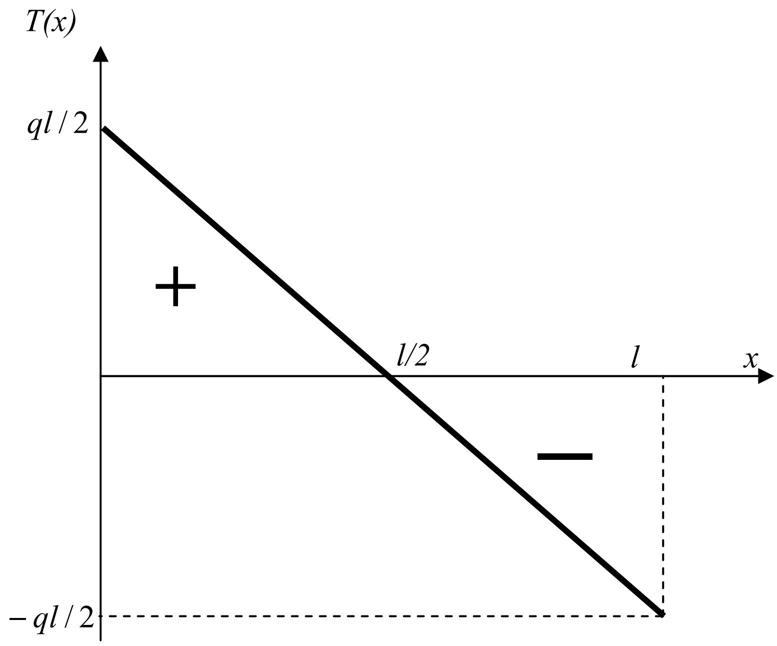
$$\sum F = 0 \Rightarrow R_A + R_B - \int_0^l q(x)dx = 0 \Rightarrow R_A + R_B = ql$$

$$\sum M /_A = 0 \Rightarrow \int_0^l q(x)xdx - R_B \times l = 0 \Rightarrow R_B = \frac{ql}{2} \Rightarrow RA = ql - R_B = \frac{ql}{2}$$

Trenchant force:

$$0 < x < l \Rightarrow T(x) = R_A - \int_0^x q(x)dx \Rightarrow T(x) = \frac{ql}{2} - qx$$

The shear force diagram is shown in the below figure:



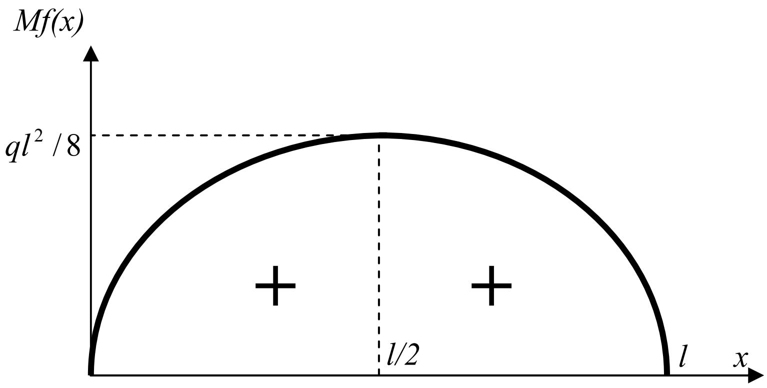
Bending moment:

$$0 < x < l \Rightarrow Mf(x) = R_A \times X - \int_0^x q(x)xdx \Rightarrow Mf(x) = \frac{ql}{2}x - \frac{q}{2}x^2$$

Or

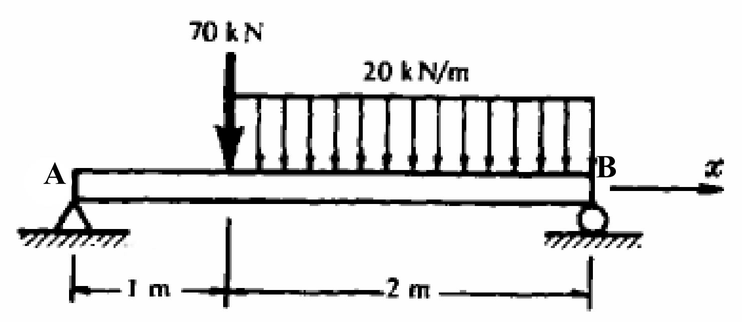
$$0 < x < l \Rightarrow Mf(x) = R_A \times X - \left[\left(\int_0^x q(x)dx\right) \times \left(\frac{1}{2}(x-0)\right)\right] \Rightarrow Mf(x) = \frac{ql}{2}x - \frac{q}{2}x^2$$

The bending moment diagram is shown in the below figure:



The maximum bending moment Mf_{max} is located at a distance of *l/2* from the support A, it is equal to $ql^2/8$.

Exemple 2:



Static equilibrium equations:

$$\sum F = 0 \Rightarrow R_A + R_B - 70 - \int_1^3 q(x)dx = 0 \Rightarrow R_A + R_B - \int_1^3 20dx = 0 \Rightarrow R_A + R_B = 110kN \; .... \; eq(1)$$

$$\sum M/_A = 0 \Rightarrow +(70*1) + \int_1^3 q(x)xdx - R_B*3 = 0 \Rightarrow +70 + \int_1^3 20xdx - R_B*3 = 0 \Rightarrow R_B = 50kN$$

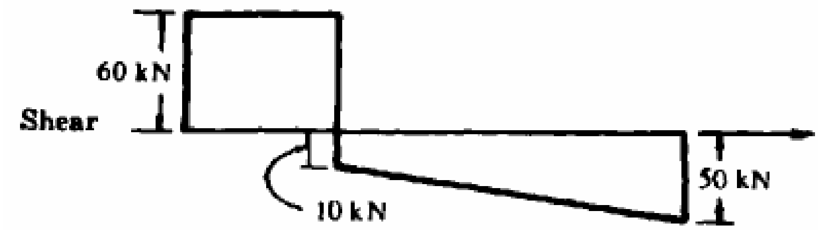
$$eq(1) \Rightarrow R_A = 110 - 50 = 60kN$$

Trenchant force:

$$0 < x < 1 \Rightarrow T(x) = R_A = 60kN$$

$$1 < x < 3 \Rightarrow T(x) = R_A - 70 - \int_1^x q(x)dx \Rightarrow T(x) = -10 - \int_1^X 20dx = -20x + 10$$

The shear force diagram is shown in the below figure:



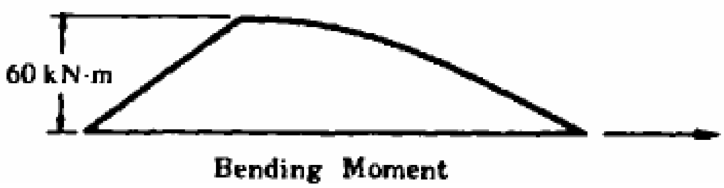
Bending moment:

$$0 < x < 1 \Rightarrow Mf(x) = R_A.x = 60x(kN.m)$$

$$1 < x < 3 \Rightarrow Mf(x) = R_A.x - 70(x-1) - \left[\left(\int_1^x q(x)dx\right) \times \left(\frac{x-1}{2}\right)\right]$$

$$\Rightarrow Mf(x) = 60x - 70x + 70 - 20 \times \left(\frac{(x-1)^2}{2}\right) = -10x^2 + 10x + 60$$

The bending moment diagram is shown in the below figure:

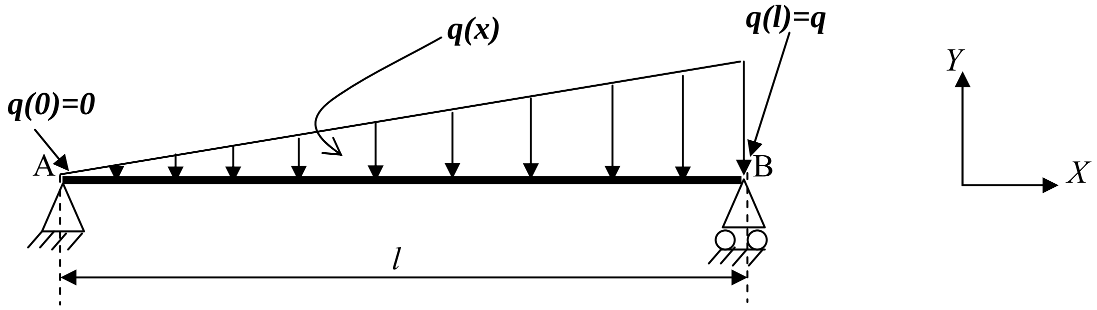


The maximum bending moment Mf_{max} is located at a distance of 1m from the support A, it is equal to 60 kN.m.

6. Non uniformly distributed load

A beam solicited by a non-uniformly distributed load means that it supports a variable linear load along its length (i.e. a variable load divided by the length of the beam). Uniformly varying load as for example triangular distributed load or trapezoidally distributed load can be presented by a linear polynomial *q(x)*. Another type of non uniformly distributed load exist which is irregular varying load.

Example 1:

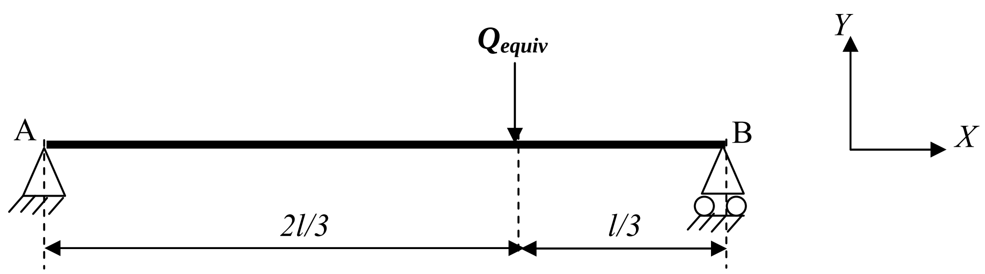


$$q(x) = \frac{q}{l}x$$

Static equilibrium equations:

$$\sum F = 0 \Rightarrow R_A + R_B - \int_0^l q(x)dx = 0 \Rightarrow R_A + R_B = \frac{q}{l}\int_0^l xdx = \frac{1}{2}ql = Q_{equiv} \quad .....eq(1)$$

The equivalent system of our above system is as follows:



$$\sum M/_A = 0 \Rightarrow Q_{equiv}\frac{2}{3}l - R_B \times l = 0 \Rightarrow R_B = \frac{2}{3}\frac{Q_{equiv}}{l}l \Rightarrow R_B = \frac{1}{3}ql$$

$$eq(1) \Rightarrow R_A = Q_{equiv} - R_B = \frac{1}{6}ql$$

Trenchant force:

$$0 < X < l \Rightarrow T(X) = RA - \int_0^X q(x)dx \Rightarrow T(X) = -\frac{1}{2}\frac{q}{l}X^2 + \frac{ql}{6}$$

x	$-\infty$	**0**	$+\infty$
T'(x)	+	**0**	−
T(x)	$-\infty$ ↗	***T(0)=T_max=ql/6***	↘ $+\infty$

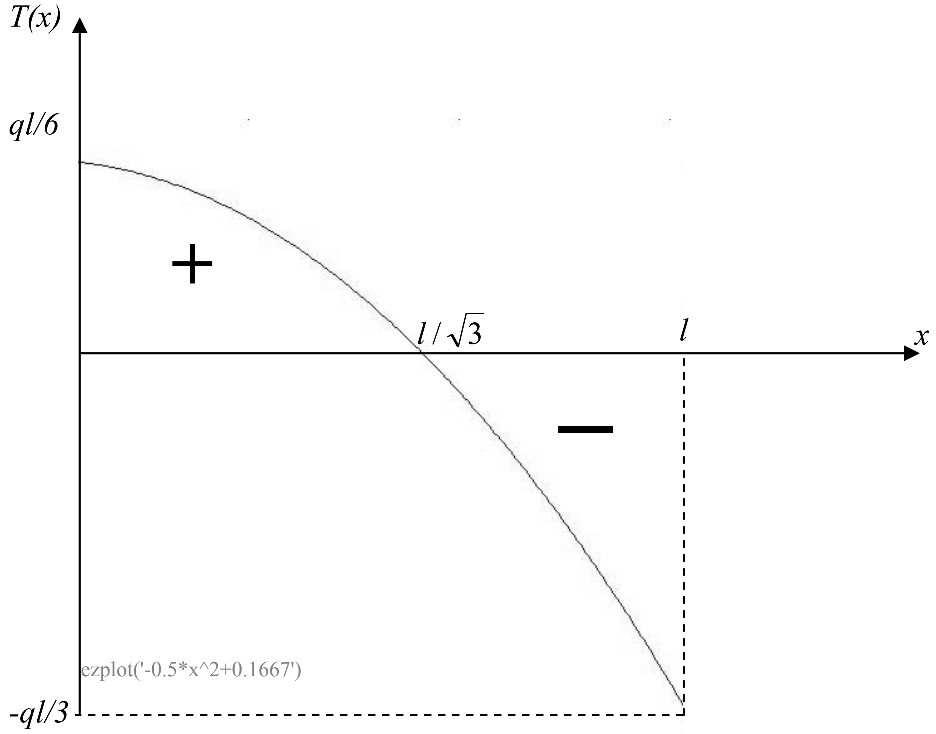
$$T'(x) = -\frac{q}{l}x = 0 \Rightarrow x = 0$$

$$T(x) = -\frac{1}{2}\frac{q}{l}x^2 + \frac{ql}{6} \Rightarrow T(0) = \frac{ql}{6}$$

$$T(x) = -\frac{1}{2}\frac{q}{l}x^2 + \frac{ql}{6} = 0 \Rightarrow x_1 = -\frac{l}{\sqrt{3}} \ ; \ x_1 = +\frac{l}{\sqrt{3}}$$

$$T(l) = -\frac{ql}{3}$$

The shear force diagram plotted by Matlab [6] is shown in the below figure:



Bending moment:

$$0 < x < l \Rightarrow Mf(x) = R_A \times X - \left[\left(\int_0^X q(x)dx\right) \times \left(\frac{1}{3}(x-0)\right)\right] \Rightarrow Mf(x) = \frac{ql}{6}x - \frac{q}{6l}x^3$$

x	$-\infty$		$-l/\sqrt{3}$		$+l/\sqrt{3}$		$+\infty$
Mf'(x)=T(x)		**−**	**0**	**+**	**0**	**−**	
Mf(x)	$+\infty$	↘	$-\frac{1}{9\sqrt{3}}ql^2$	↗	$+\frac{1}{9\sqrt{3}}ql^2$	↘	$-\infty$

$$Mf'(x) = T(x) = -\frac{1}{2}\frac{q}{l}x^2 + \frac{ql}{6}$$

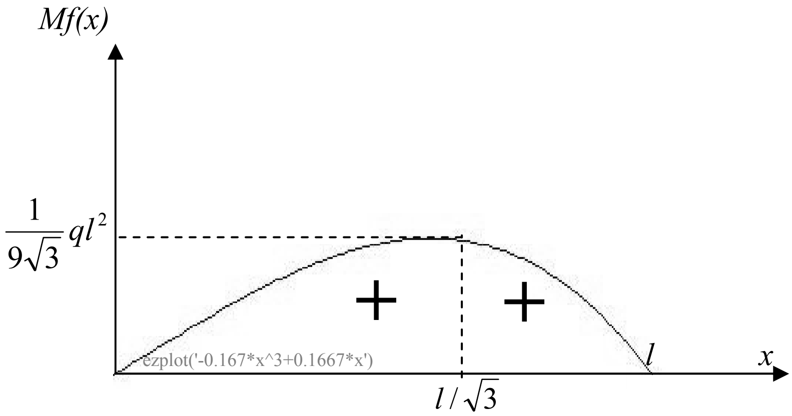
$Mf''(x) = T'(x) = -\frac{q}{l}X = 0 \Rightarrow x = 0$ is the deviation point.

$$Mf(X) = \frac{ql}{6}x - \frac{q}{6l}x^3 \Rightarrow Mf(0) = 0$$

$$Mf(X) = \frac{ql}{6}x - \frac{q}{6l}x^3 = 0 \Rightarrow x\left(\frac{ql}{6} - \frac{q}{6l}x^2\right) = 0 \Rightarrow x_1 = 0;\ x_2 = l;\ x_3 = -l$$

$$Mf(l) = 0$$

The bending moment diagram plotted by Matlab [6] is shown in the below figure:



The maximum bending moment Mf_{max} is located at a distance of $l/\sqrt{3}$ from the support A, it is equal to $\frac{1}{9\sqrt{3}}ql^2$.

Numerical application with *RDM6*:

Using the *RDM6* software [7], we could plot the variation of the shear force, the bending moment, the normal stress due to bending and we can also extract the beam deflection.

If we take for the previous example, a length l equal to 1m, a radius R of the beam equal to 10 mm and a load q equal to 1000 N/m. The distance from the support A where Mf_{max} is located is equal to $l/\sqrt{3} = 0{,}578$ m; we then obtain with our previous theoretical calculation a maximum bending moment equal to:

$Mf_{\max} = \frac{1}{9\sqrt{3}} ql^2$, so after calculation, we find Mf_{max}= 64.15 N.m.

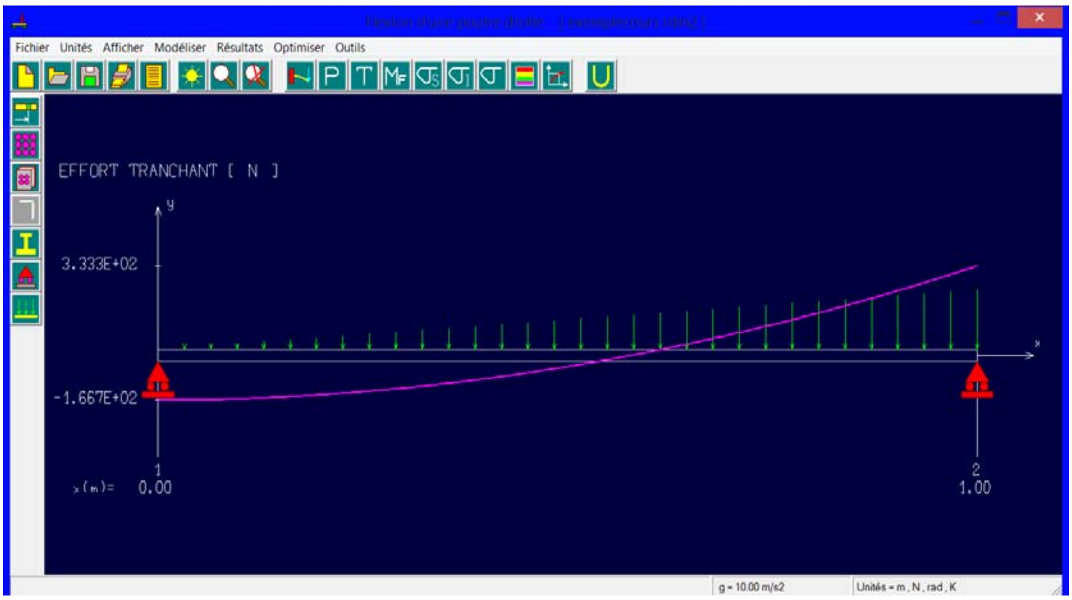
Using formula (I-4) which will be seen later, the maximum stress is given by the following relationship:

$$\sigma_{\max} = \frac{Mf_{\max} * y_{\max}}{I_z} = \frac{\left(\frac{1}{9\sqrt{3}} ql^2\right)(R/2)}{\left(\pi R^4 / 4\right)} = 40{,}85 \text{ Mpa} = 4{,}085\ 10^7 \text{ Pa.}$$

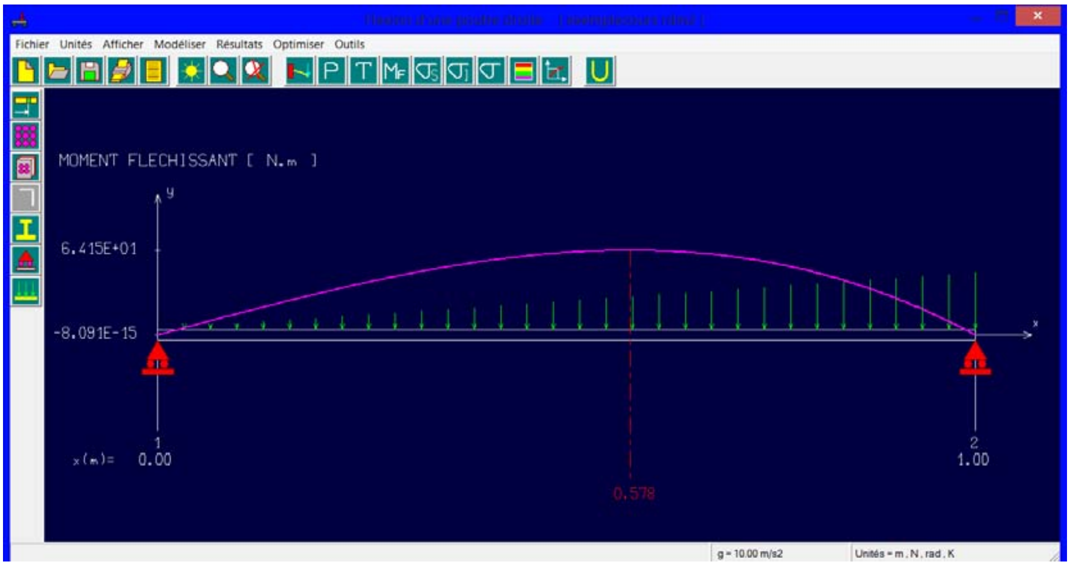
Note that this stress is located at the same point of the beam where the maximum bending moment was located.

A very good agreement was obtained between the theoretical results and the numerical results obtained by the *RDM6* finite element software. Here are some results obtained by *RDM6*.

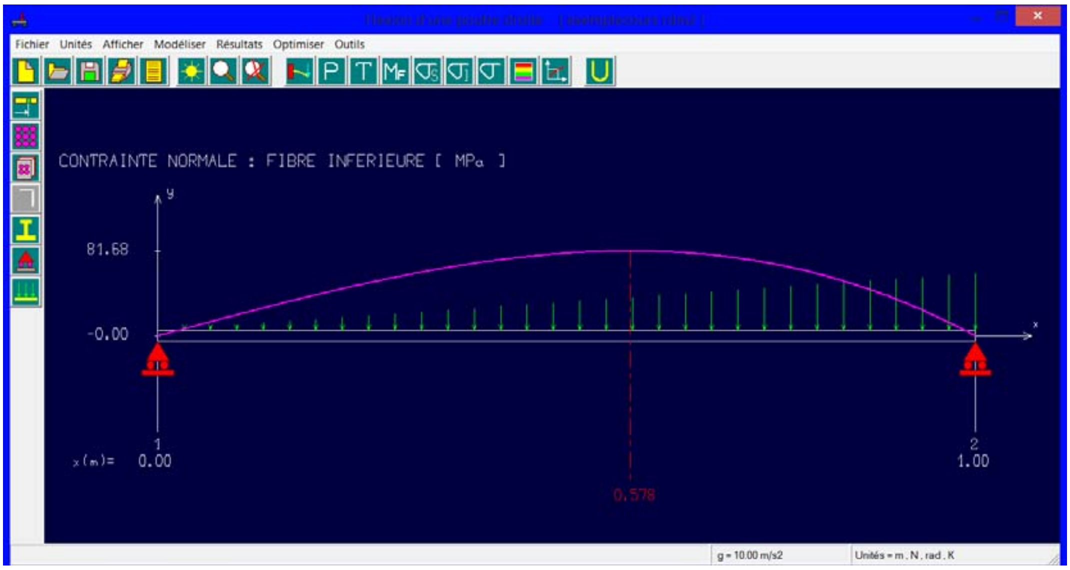
Variation along the beam of the shear force obtained by *RDM6*:



Variation along the beam of the bending moment obtained by *RDM6*:



Variation along the beam of the normal stress obtained by *RDM6*:



Deflection of the beam obtained by *RDM6*:

Rapport de calcul RDM6

```
+-----------------------------+
| Flexion d'une poutre droite |
+-----------------------------+
Utilisateur : BELOUFA
Nom du projet : C:\Users\BELOUFA\Documents\exemplecours rdm2
Date : 18 septembre 2023
+---------------------+
| Données du problème |
+---------------------+
+-----------+
|  Matériau |
+-----------+
Nom du matériau = Acier
Module de Young = 210000 MPa
Masse volumique = 8000 kg/m3
Limite élastique = 250 MPa
+---------------+
|  Noeuds [ m ] |
+---------------+
Noeud  1 :  x =   0.000
Noeud  2 :  x =   1.000
+-----------------------+
|  Section(s) droite(s) |
+-----------------------+
Noeuds 1 --> 2
  Rond plein : D = 20.00 (mm)
  Aire = 3.14 cm2
  Moment quadratique : Iz = 0.79 cm4
  Fibre supérieure : vy = 10.00 mm   Wel.z = 0.79 cm3
  Fibre inférieure : vy = 10.00 mm   Wel.z = 0.79 cm3
Poids de la structure = 25.13 N (g = 10.00 m/s2)
+-----------------------+
|  Liaison(s) nodale(s) |
+-----------------------+
Noeud  1 : Flèche = 0
Noeud  2 : Flèche = 0
+-------------------+
|  Cas de charge(s) |
+-------------------+
Charge linéairement répartie : Noeuds = 1 -> 2    pyo = 0.00   pye = -1000.00 N/m
+-----------+
| Résultats |
+-----------+
+---------------------------------+
| Déplacements nodaux [ m , rad ] |
+---------------------------------+
Noeud    Flèche     Pente
  1    0.000000  -0.011789
  2    0.000000   0.013473
Dy maximal = 0.00000E+00 m à x = 0.000 m
Dy minimal = -3.95442E-03 m à x = 0.520 m
+------------------------------------+
| Efforts intérieurs [ N  N.m  MPa ] |
+------------------------------------+

Ty = Effort tranchant    Mfz = Moment fléchissant   Sxx = Contrainte normale

Noeud       Ty          Mfz         Sxx

  1      -166.67       -0.00       -0.00
  2       333.33        0.00        0.00

Moment flechissant maximal = 64.15 N.m à 0.578 m
Moment flechissant minimal = -0.00 N.m à 1.000 m

Contrainte normale maximale = 81.68 MPa à 0.578 m
Contrainte normale minimale = -81.68 MPa à 0.578 m
+---------------------------------+
| Action(s) de liaison [ N  N.m ] |
+---------------------------------+
Noeud  1   Fy =   166.67
Noeud  2   Fy =   333.33
+----------------------------+
| Informations sur le calcul |
+----------------------------+
Pivot minimal =  4.9480084294011OE+0003
```

7. Normal tensile and compressive stresses due to the bending moment in a beam

All the definitions and examples shown previously to calculate the variation of the bending moment in the beam are useful now to determine the variation of the mechanical stress along the beam. The knowledge of the variation of the mechanical stress in the beam is necessary to find the dangerous section in the beam which corresponding to the maximum bending moment or the maximum stress; this latter is compared to the yield stress of the material or to the material allowable stress in order to see if the beam will be fractured or not. If we notice that the beam does not resist to the loading that we have applied to it, then, many solutions can be deployed as: resize the beam again, change it form, use another resistant material, decrease the loading magnitude or change the place where this loading is applied, add more supports, etc.

The figure below shows the normal tensile and compressive stresses generated in a beam subjected to bending.

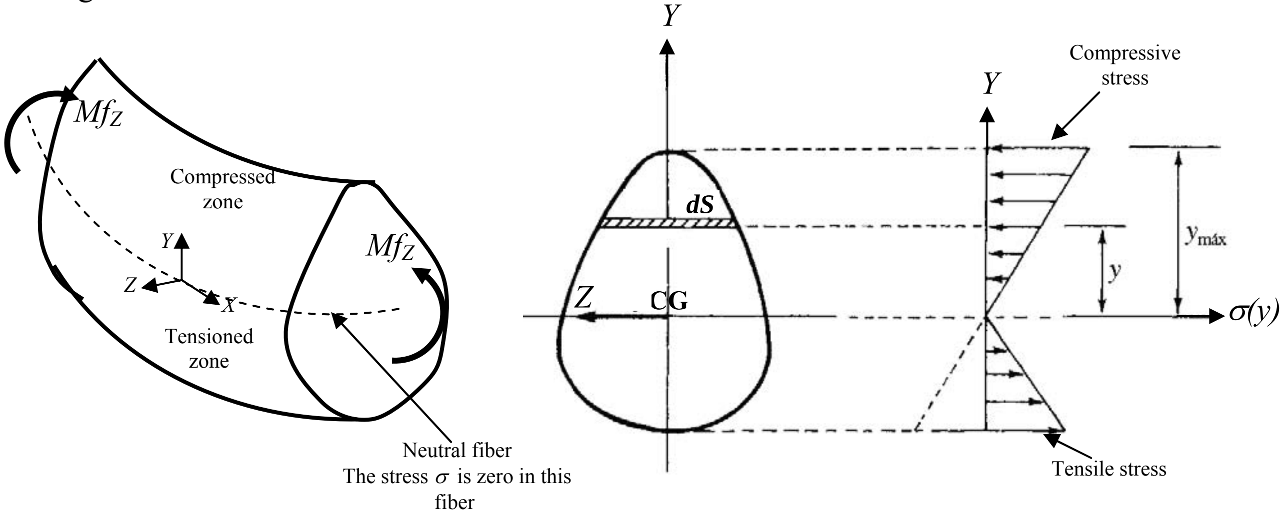


Figure I- 13: Normal tensile and compressive stresses in a beam loaded in bending around the *Z* axis

Note that the axis of the neutral axis always passes through the axis of the gravity center or through the centroid of the cross-sectional area; y_{max} is the farthest distance from the neutral fiber axis (Figure I- 13).

The normal stress for any variable *x* and *y* is given by following formula (formula which will be demonstrated later in the Chapter II):

$$\sigma(x,y) = \frac{Mf(x) \times y}{I_z} \quad \text{(I-3)}$$

The maximum normal stress due to bending is equal to:

$$\sigma_{\max} = \frac{Mf_{\max} \times y_{\max}}{I_z} \quad \text{(I-4)}$$

If we take our example illustrated previously in paragraph 4.1 or in the below figure, we obtained a maximum bending moment Mf_{max} equal to Wl, thus the inertia moment of the beam cross-sectional is equal to $I_z = bh^3/12$. Thus, since the bending takes place around the Z axis, and according to the arrangement of the beam presented in the figure below, the distance y_{max} will then be equal for our case to $h/2$.

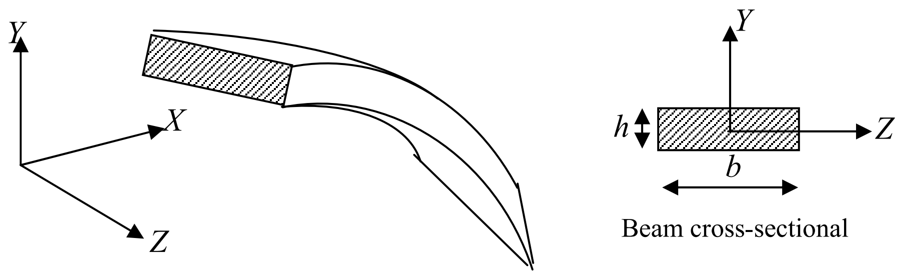


Figure I- 14: The deflected shape of a cantilever beam with it cross-sectional

Let us now apply the formula (I-4); we obtain a maximum stress equal to:

$\sigma_{\max} = \dfrac{6Wl}{bh^2}$ in N/m^2 ou Pa.

The calculated maximum stress will be compared either to the yield stress σ_Y of the beam material or to the admissible or allowable stress of the beam material which is calculated by the following formula:

$$\sigma_{adm} = \frac{\sigma_Y}{Cs} \qquad \text{(I-5)}$$

With Cs is the safety factor or coefficient. Cs is between 2 and 4 for normal constructions and it is ≥ 10 for constructions which would endanger the lives of people.

If $\sigma_{\max} > \sigma_{adm}$, then the beam will break or will undergo plastic deformation, otherwise, the beam will undergo elastic deformation.

We give in the below table (Table I-2), some essential mechanical properties (including the yield elastic limit σ_Y) for some metallic materials:

	Specific weight		Young's modulus		Yield and ultimate Stress		Coefficient of linear thermal expansion		
Material	lb/in^3	kN/m^3	lb/in^2	GPa	σ_Y (MPa)	σ_U (MPa)	10*e*–6/°F	1 *e*–6/°C	Poisson's ratio
I. Metals in slab, bar, or block form									
Aluminum alloy	0.0984	27	10–12*e*6	70–79	100-500	310–550	13	23	0.33
Brass	0.307	84	14–16*e*6	96–110	70-550	300–590	11	20	0.34
Copper	0.322	87	16–18*e*6	112–120	200 [9]	230–380	9.5	17	0.33
Nickel	0.318	87	30*e*6	210	140-620	310–760	7.2	13	0.31
Steel	0.283	77	28–30*e*6	195–210	200-1700	550–2000	6.5	12	0.30
Titanium alloy	0.162	44	15–17*e*6	105–120	760-900	900–970	4.5–5.5	8–10	0.33

Table I- 2: Properties of common engineering materials at 20°C [3] & [8] [3]

8. Tangential stress or shear stress due to the shear force in a beam

Due to the bending moment *Mfz*, a shear force *T(x)* (Figure I- 15) and a normal stress σ will be created. F_N being the normal force corresponding to the *x* axis, due to the normal stress σ acting on the near face; $F_F = F_N + dF_N$ is due to the normal stress $\sigma + d\sigma$ acting on the far face; and dF_B is the force due to the shear stress τ acting on the bottom face.

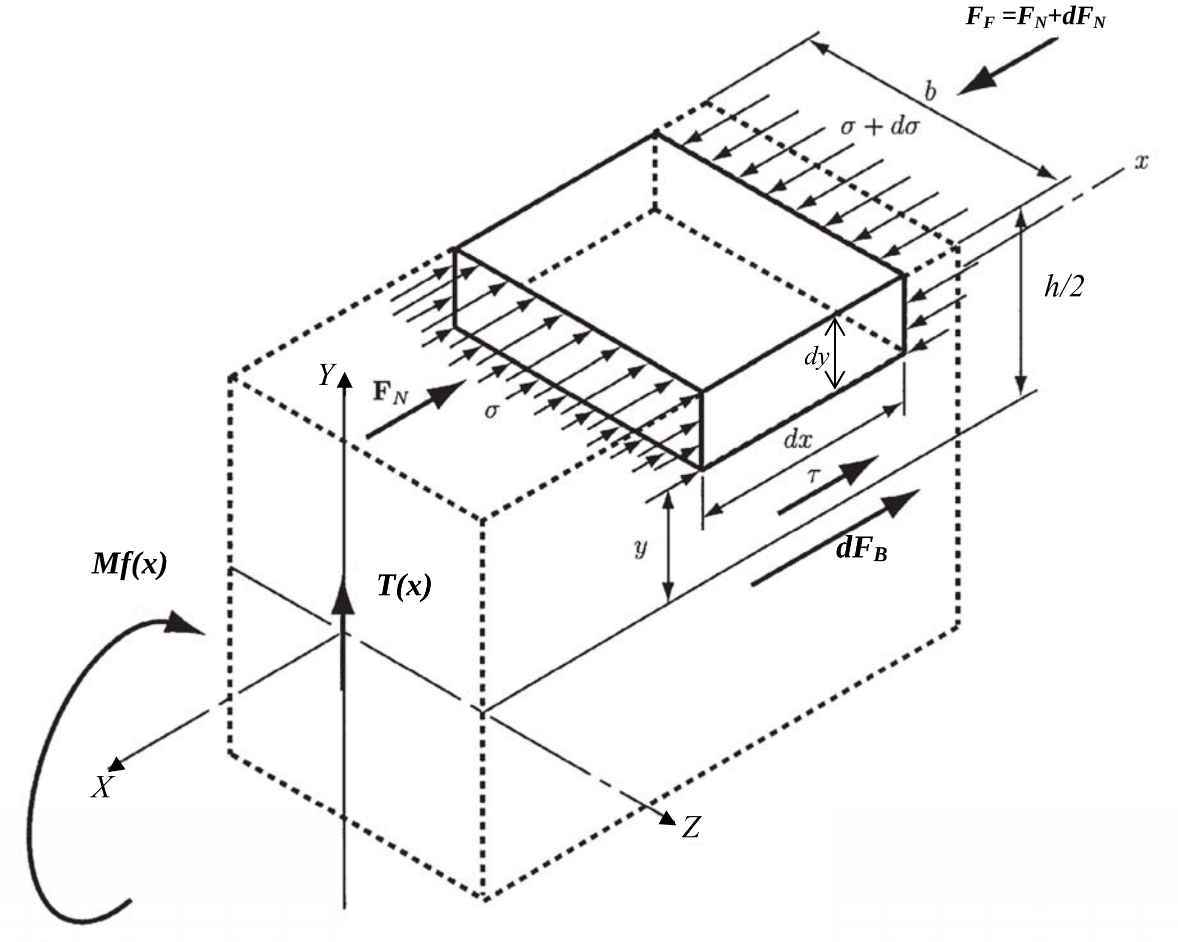


Figure I- 15: Shear stresses in the beam [10]

The normal force F_N can be calculated by:

$$F_N = \int_y^{h/2} \sigma \, dS = b \int_y^{h/2} \sigma \, dy \tag{I-6}$$

As we have indicated in the equation (I-3), $\sigma = Mf . y / I_z$), equation (I-6) can be written as:

$$F_N = \frac{Mf . b}{I_z} \int_y^{h/2} y \, dy \tag{I-7}$$

So, we can deduce dF_N:

$$dF_N = \frac{dMf . b}{I_z} \int_y^{h/2} y \, dy \tag{I-8}$$

The force dF_B due to the shear stress τ is given by:

$$dF_B = \tau\, dS' = b\tau\, dx \tag{I-9}$$

The forces equilibrium gives:

$$\Sigma F = 0 \Rightarrow F_N + dF_B - F_F = 0 \Rightarrow F_N + dF_B - F_N - dF_N = 0 \Rightarrow dF_B = dF_N \tag{I-10}$$

Substituting (I-8) and (I-9) in (I-10), we can obtain:

$$b\tau\, dx = \frac{b\, dMf}{I_z} \int_y^{h/2} y\, dy \tag{I-11}$$

$$b\tau\, dx = \frac{b\, dMf}{I_z} \int_y^{h/2} y\, dy$$

$$\Rightarrow \tau\,(\mathrm{x}) = \frac{dMf(x)}{I_z dx} \int_y^{h/2} y\, dy \text{ , avec } T(x) = \frac{dMf(x)}{dx}$$

$$\Rightarrow \tau\,(x,y) = \frac{dMf(x)}{I_z dx} \int_y^{h/2} y\, dy = \frac{1}{2}\frac{T(x)}{I_z}\left[(h/2)^2 - y^2\right] \text{ avec } I_z = \frac{bh^3}{12} \tag{I-12}$$

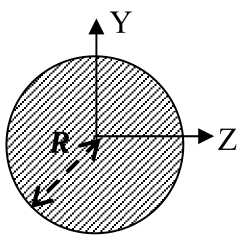
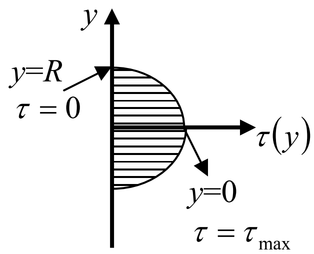
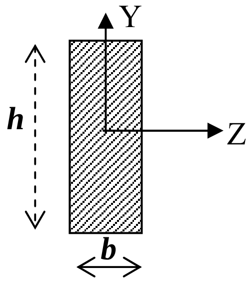
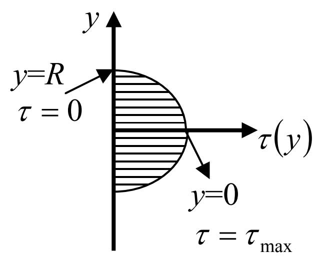
Cross-sectional type	**Curves of $\tau(y)$**	**Shear stresses**
		$\tau(x,y) = \frac{1}{3}\frac{T(x)}{I_z}\left[R^2 - y^2\right]$ $I_z = \frac{\pi R^4}{4}$ $\tau_{mean} = \frac{T}{S} = \frac{T}{\pi R^2}$ $\tau_{\max} = \frac{4}{3}\tau_{mean}$
		$\tau(x,y) = \frac{1}{2}\frac{T(x)}{I_z}\left[(h/2)^2 - y^2\right]$ $I_z = \frac{bh^3}{12}$ $\tau_{mean} = \frac{T}{S} = \frac{T}{bh}$ $\tau_{\max} = \frac{3}{2}\tau_{mean}$

Table I- 3: Mean and maximal shear stresses for circular and rectangular cross-sectional of beams

Directed works No. 1 “Pure bending of symmetrical beams”

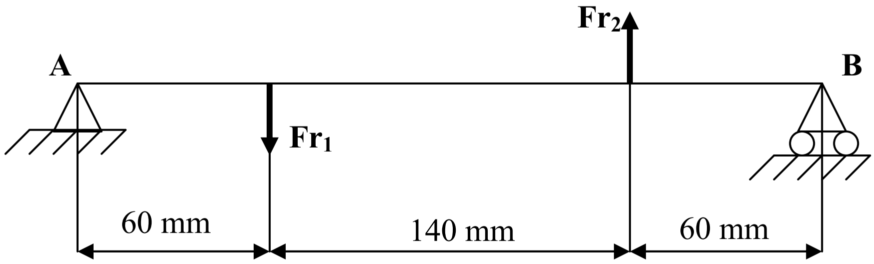
Exercise N°1

The beam presented below is supported at its both ends A and B and it is subjected to bending by two forces Fr1 = 284 N and Fr2 = 852 N.

1) Calculate the vertical reactions at the supports A and B.

2) Plot the variation of the shear force over the entire length of the beam.

3) Plot the variation of the bending moment along the beam and determine the maximum value of the bending moment.

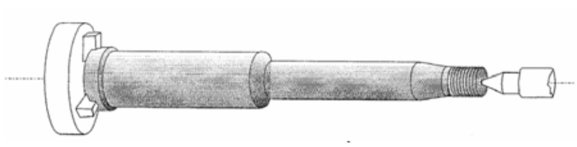


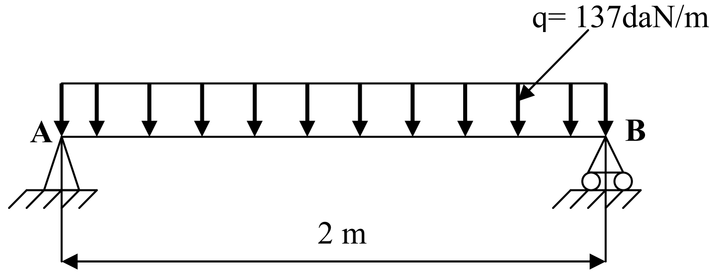
Exercise N°2

The beam shown in the figure below is mounted on a lathe machine, the material of the beam is steel with a circular section (diameter d=15 cm), the linear weight of the beam is 137 daN/m, the length of the beam is L = 2m, the practical strength or allowable stress of the steel is σ_{adm}=200 MPa. The beam to be machined can be modeled as a beam placed on two supports A and B and stressed by its own weight which is uniformly distributed over its entire length.

1) Plot the variation of the shear force over the entire length of the beam.

2) Plot the variation of the bending moment along the beam and determine the maximum value of the bending moment. Calculate the maximum bending stress.





Exercise N°3

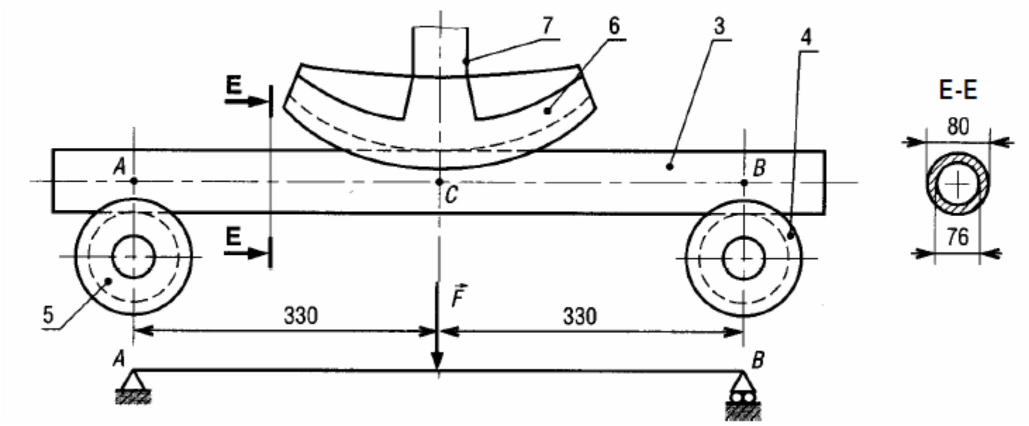
The device shown in the figure below makes it possible to bend a hollow tube (3), the bending force F applied by the bending head (7) is provided by a hydraulic cylinder, the tube is placed on two rollers (4) and (5). The yield stress of the tube material is σ_Y=340 N/mm² or MPa.

1) Determine the maximum shear force in the tube.

2) Calculate the average shear stress in the middle of the tube, the dimensions of the tube section are shown in the figure below.

3) Calculate the maximum normal stress generated at the tube as a function of the bending force.

4) Then determine the effort required to bend the tube.



P.S: Dimensions are given in mm.

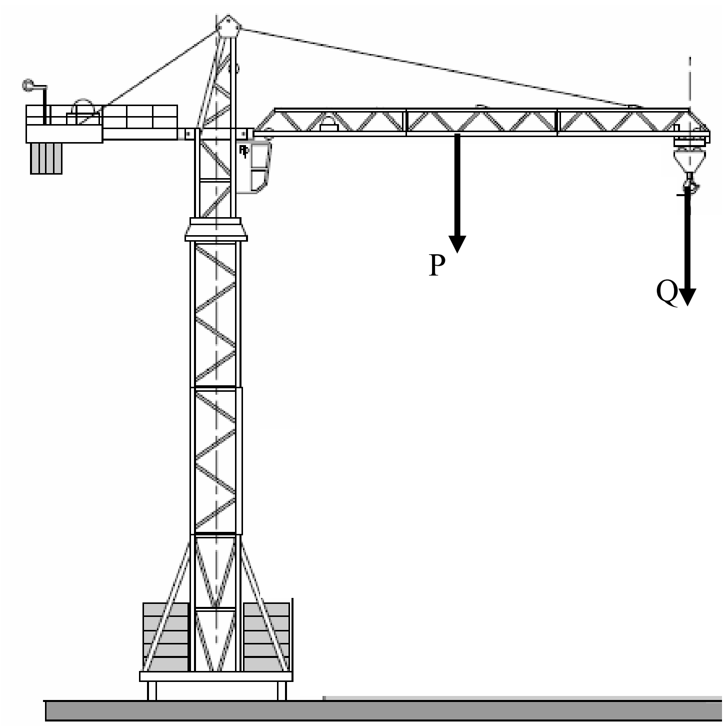
Exercise N°4

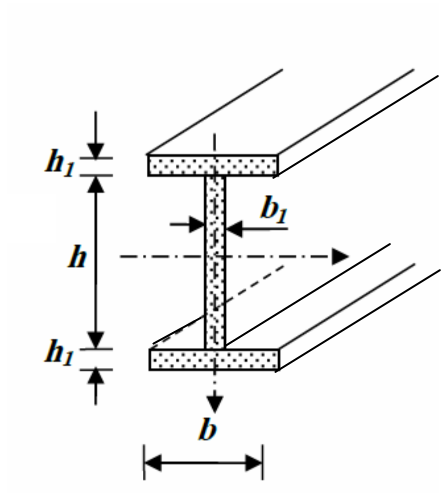
A tower crane jib is made up of two IPN type beams. The weight P of each beam is equal to 0.5 tones and the load to be lifted Q weighs is 2 tones, the material of the beams is steel with an elastic limit σ_y = 800 MPa. The safety coefficient *Cs* is equal to 2.

1) Calculate the moment of inertia I of the section of the IPN beam.

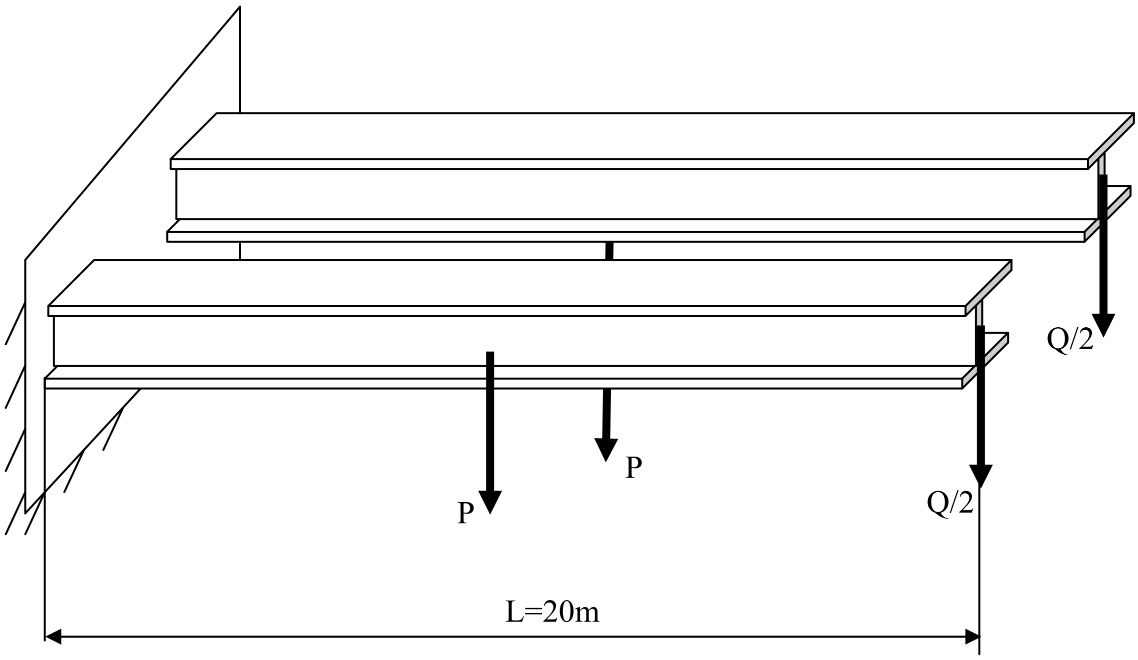
2) Determine the maximum bending moment in each beam.

3) Calculate the maximum stress, do you think these beams will hold this weight Q.





h_1=b_1=5 mm
h=b=200 mm



Chapter II
Deflection of symmetrical beams for pure bending

Chapter II: Deflection of symmetrical beams for pure bending

1. Demonstration of the normal stress calculation in flexure

The engineer is interested not only in the stresses caused by loads on a beam but also in the deflection produced by these loads. It is sometimes mentioned that you should not exceed a certain value of the maximum deflection. In the figure below (Figure II- 1), before the deformation of the beam, all the top, neutral and bottom surfaces are equal, but when we apply a constant moment on the beam, the highest compressed surface *mp* will bend downward with a distance less than that of the neutral surface *nn'* (the latter will not undergo any deformation), and the tense surface *m'p'* will bend downward with a distance greater than that of the neutral surface *nn'*, this will give the appearance of an angle $d\theta$. Therefore, *r* is the radius of curvature of the neutral fiber.

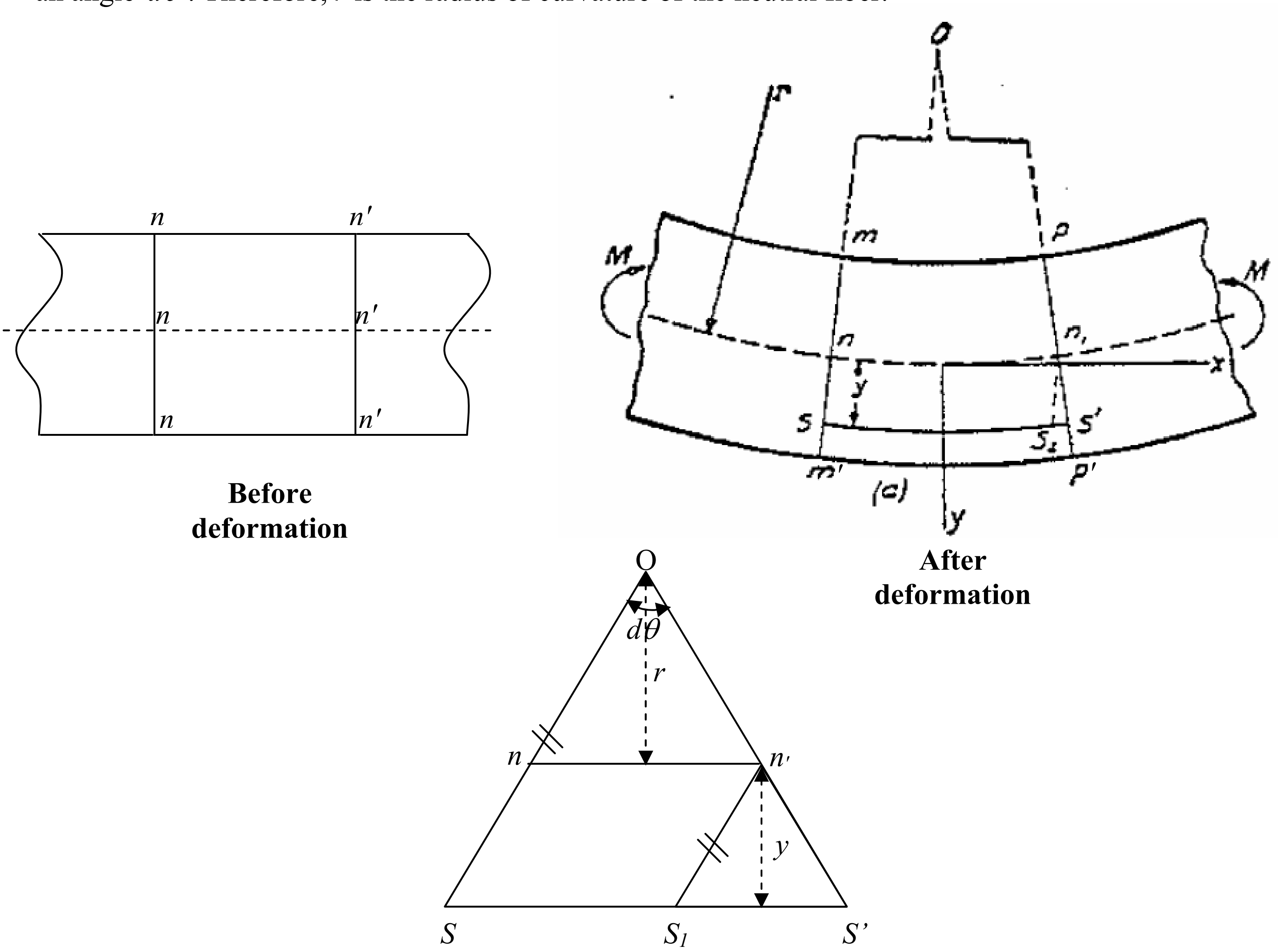


Figure II- 1: Normal stress in flexure

nn' $=SS_1$ = initial length before deformation.

$S'S_1$ is the elongation of the fiber along the X axis which is located at a distance of *y* from the neutral axis.

We have:

$$tg(d\theta/2)=\frac{nn'}{2r}=\frac{S'S_1}{2y}\Rightarrow\frac{S'S_1}{nn'}=\frac{y}{r}=\frac{\Delta l}{l_0}=\varepsilon_{xx} \quad \text{(II-1)}$$

Referring to the below figure, we have:

$$\sigma_{xx}=E\varepsilon_{xx}=E\frac{y}{r}=\frac{dF_x}{dA} \quad \text{(II-2)}$$

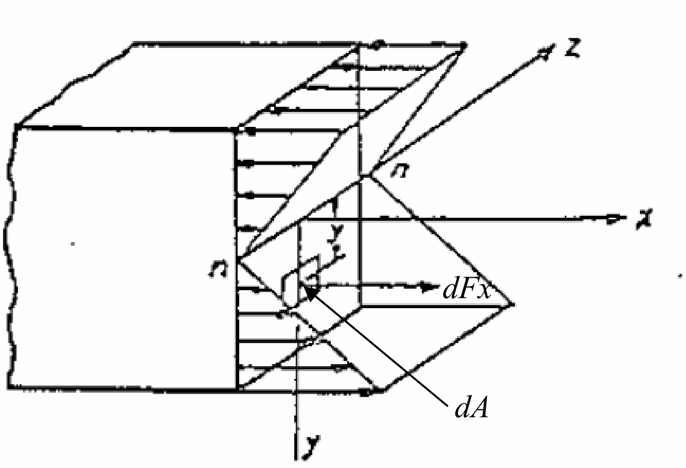


Figure II- 2: Normal force in flexure Erreur ! Source du renvoi introuvable.

Since the moment of the normal and the tensile forces about the neutral axis is zero, then, their sum is equal to zero (the sum is represented by the integral form):

$$\sum dFx=\int dFx=0\Rightarrow Fx=0 \quad \text{(II-3)}$$

$$\Rightarrow F_x=\int\sigma_{xx}dA=\int E\frac{y}{r}dA=0 \quad \text{(II-4)}$$

The bending moment *Mf* is equal to:

$$Mf=F_x y=\iint\sigma_{xx}ydA=\iint E\frac{y}{r}ydA=\iint E\frac{y^2}{r}dA=\frac{E}{r}\iint y^2dA \quad \text{(II-5)}$$

$$I_z=\iint y^2dA \quad \text{(II-6)}$$

$$\Rightarrow Mf=\frac{EI_z}{r} \quad \text{(II-7)}$$

$$\Rightarrow\frac{1}{r}=\frac{Mf}{EI_z} \quad \text{(II-8)}$$

Knowing that $\frac{1}{r}=\frac{Mf}{EI_z}$, then:

$$\sigma_{xx}=E\varepsilon_{xx}=E\frac{1}{r}y=E\frac{Mf}{EI_z}y=\frac{Mf.y}{I_z} \quad \text{(II-9)}$$

$$\sigma_{xx}(x,y)=\frac{Mf(x).y}{I_z} \quad \text{(II-10)}$$

2. Deflection of beams with constant cross-section

The below figure shows the deformed shape of a beam subjected to bending.

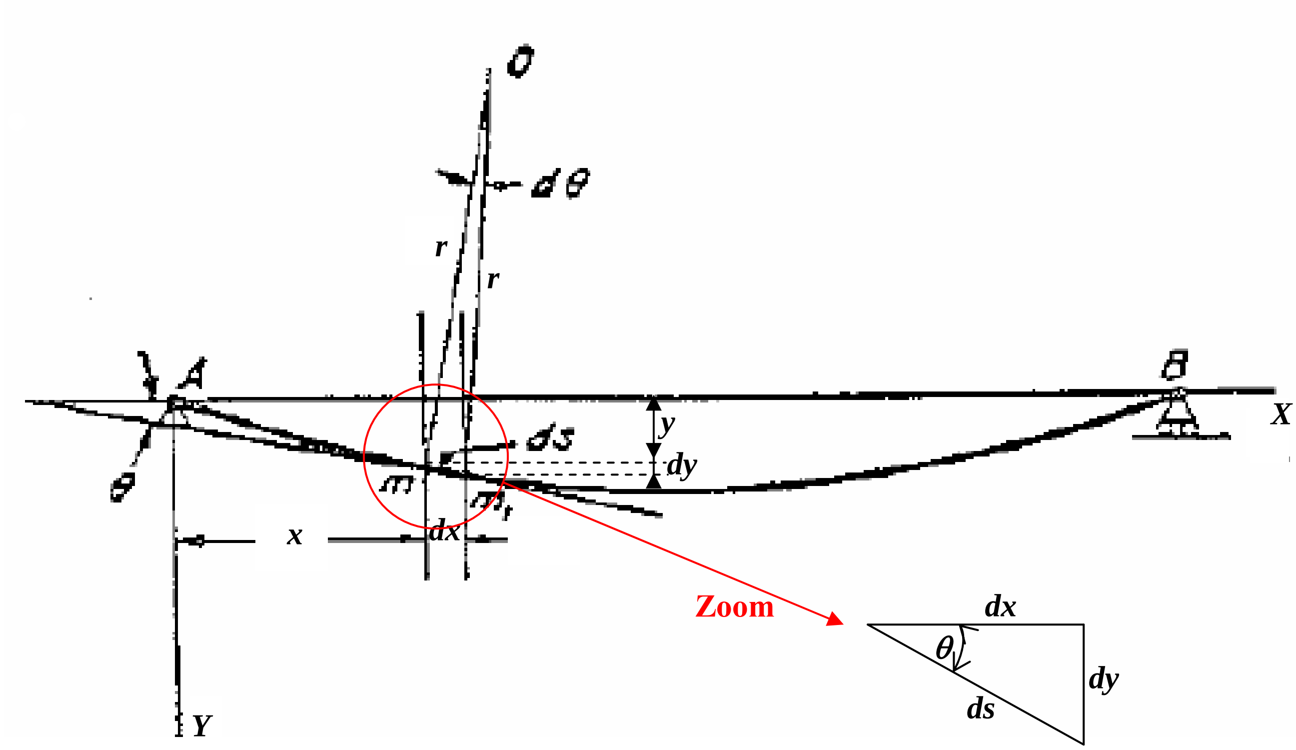


Figure II- 3: Deflection of a beam

We take note that $ds \approx dx$. Indeed, in practice we only tolerate very small deflections, so the curvature of the beam must remain almost flat and the angle $d\theta$ will be too small, we will then have:

$$tg\theta \approx \theta = \frac{dy}{dx} \tag{II-11}$$

y is the deflection of the beam at x point.

Knowing that:

$$tg(d\theta) \approx d\theta = ds / r = dx / r \Rightarrow 1/r = d\theta / dx \tag{II-12}$$

Let us substitute θ in the last equation:

$$\frac{1}{r} = \frac{d^2 y}{dx^2} = \frac{d\theta}{dx} \Rightarrow \theta(x) = \frac{dy(x)}{dx} \Rightarrow y(x) = \int \theta(x) dx \tag{II-13}$$

We know from the equation (II-8) that:

$$\frac{1}{r} = \frac{Mf}{EI_z} \tag{II-14}$$

Let's compare the two previous equations, we can write the following differential equation:

$$\frac{d^2 y(x)}{dx^2} = \frac{d\theta(x)}{dx} = \frac{Mf}{EI_z} \tag{II-15}$$

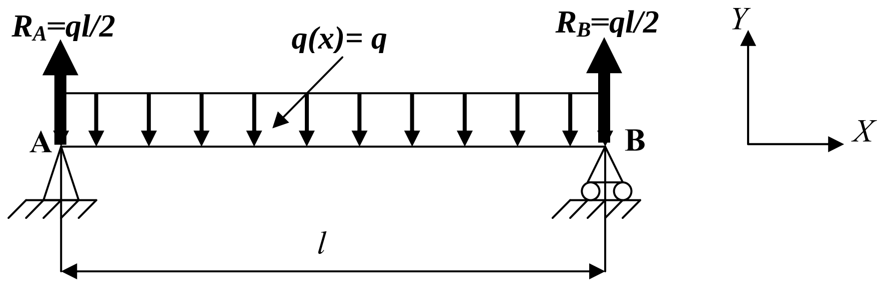
$$\Rightarrow y'' - (Mf / EI_z) = 0 \text{ or } \theta' - (Mf / EI_z) = 0 \tag{II-16}$$

When, *y(x)* represents the variation along *x*-axis of the deflected beam.

2.1. Double-integration method of the beam deflected differential equation

The calculation procedure of the beam deflection formula and the maximum beam deflection using the integration method is presented in this paragraph. Below is an example that will show the calculation of the beam deflection using this method.

Example 1:



$$0 < x < l \Rightarrow Mf(x) = R_A x - \int_0^x q(x)x dx \Rightarrow Mf(x) = \frac{ql}{2}x - \frac{q}{2}x^2$$

$$Mf = EI_z \frac{d^2y}{dx^2} \Rightarrow \frac{dy}{dx} = \theta(x) = \int \frac{Mf(x)}{EI_z}dx = \frac{1}{EI_z}\int\left(\frac{ql}{2}x - \frac{q}{2}x^2\right)dx$$

$$\Rightarrow \frac{dy}{dx} = y'(x) = \theta(x) = \frac{1}{EI_z}\left(-\frac{q}{6}x^3 + \frac{ql}{4}x^2 + C_1\right)$$

$$\Rightarrow y(x) = \int \theta(x)dx = \int \frac{1}{EI_z}\left(-\frac{q}{6}x^3 + \frac{ql}{4}x^2 + C_1\right)dx$$

$$\Rightarrow y(x) = \frac{1}{EI_z}\left(-\frac{q}{24}x^4 + \frac{ql}{12}x^3 + C_1 x + C_2\right)$$

The boundary conditions are:

At the A, B supports, we have respectively: $y(0) = y(l) = 0$

$$y(0) = 0 \Rightarrow C_2 = 0$$

$$y(l) = 0 \Rightarrow C_1 = -\frac{ql^3}{24}$$

So, $y(x) = \frac{1}{EI_z}\left(-\frac{q}{24}x^4 + \frac{ql}{12}x^3 - \frac{ql^3}{24}x\right)$

The maximum deflection $f_{\max}$ is the optimum of the previous function *y(x)*, it is equal to:

$$f_{\max} = y(l/2) = -\frac{5ql^4}{384EI_z}$$

Note:

EI_z represents the bending rigidity or the bending stiffness (modulus of elasticity or Young's modulus multiplied by the surface moment of inertia of the cross section of the beam), so if E or I_z increases, we will have a low deformation and a low deflection f.

The slope of the deformation of the beam at $x=l/2$ is zero because (dy/dx) at $l/2$ =0 and therefore the maximum deflection f_{max} is at $l/2$.

The maximum angle is found at the supports A and B, it is equal to:

$$\theta_{\max} = \left(\frac{dy}{dx}\right)_{\max} = y'(0) = y'(l) = \pm\frac{ql^3}{24EI_z}$$

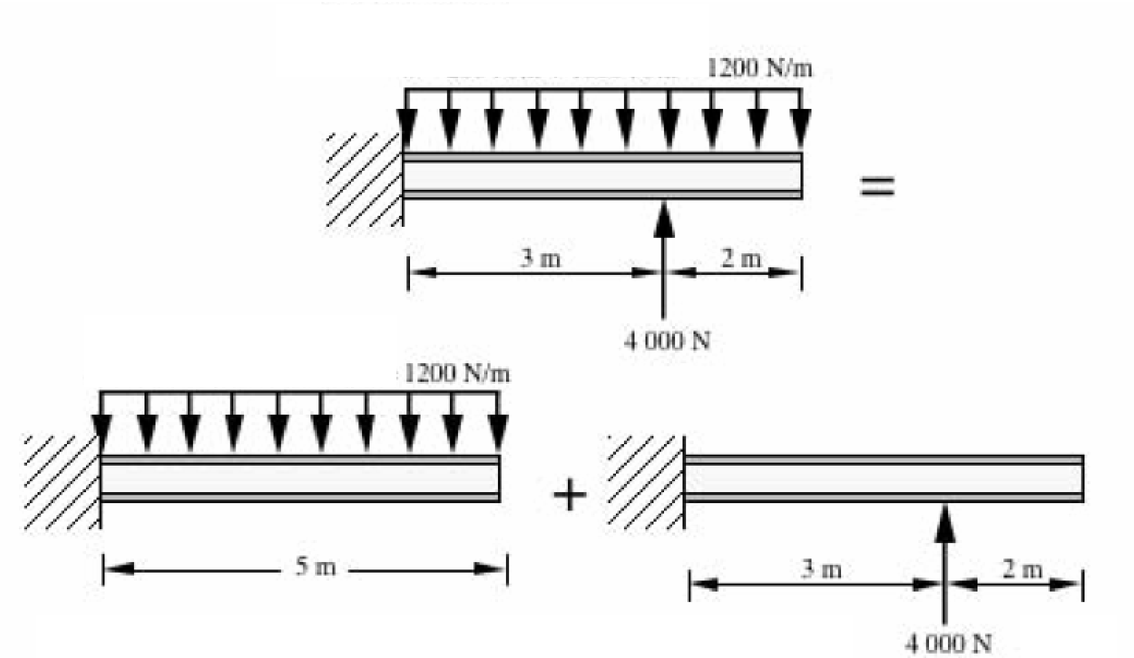
$$\Rightarrow |\theta(0)| = |\theta(l)| = \frac{ql^3}{24EI_z}$$

2.2. Method of superposition

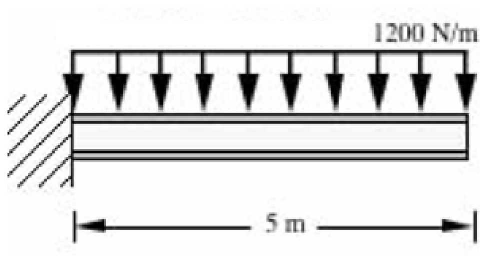
When we have a beam subjected to many loads (forces, moments, uniformly load, etc), the final slope or deflection at any point on this beam is equal to the sum of different deflections or slopes calculated separately for each load.

Example:

We want to determine by the superposition method the resultant of the maximum deflection of the steel console beam presented in the below figure: The profile has a mass *m* of 120 kg/m, which gives a weight *W* equal to 120kg *5m*g = 6000N for 5m (the gravity g is taken 10 m/s^2); therefore a uniformly distributed load of 6000/5=1200N/m will be applied to the beam, the final load q will be equal to 1200 N/m. Another force equal to 4000N is applied inversely to the distributed load q at a distance of 3m from the embedding point.



For the first beam:



$$\sum F = 0 \Rightarrow R - \int_0^5 q dx = 0 \Rightarrow R = \int_0^5 1200 dx \Rightarrow R = 6000 \text{ N}$$

$$\sum M = 0 \Rightarrow -Mc + \int_0^5 qx dx = 0 \Rightarrow Mc = \int_0^5 \frac{1200}{2} x^2 dx \Rightarrow Mc = 15000 \text{ N.m}$$

$$Mf(x) = Rx - Mc - q\frac{x^2}{2} = 6000x - 15000 - 600x^2$$

$$\frac{dy}{dx} = \int \frac{Mf(x)}{EI}dx = \frac{1}{EI}\left(-200x^3 + 3000x^2 - 15000x + C_1\right)$$

$$y_1(x) = \frac{1}{EI}\left(-50x^4 + 1000x^3 - 7500x^2 + C_1x + C_2\right)$$

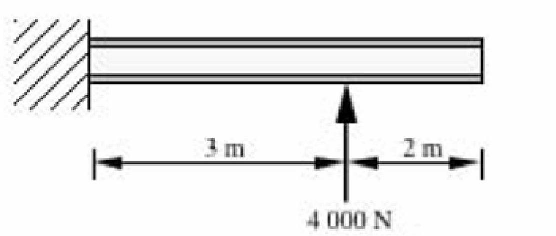
The boundary conditions are:

$$\theta(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \Rightarrow C_1 = 0 \text{ et } y(0) = 0 \Rightarrow C_2 = 0$$

So, $y_1(x) = \frac{1}{EI}\left(-50x^4 + 1000x^3 - 7500x^2\right)$

$$\left|y_1(x)\right|_{Max} = \left|y_1(5)\right|_{Max} = \frac{-93750}{EI}$$

For the second beam:



$$\sum F = 0 \Rightarrow R + 4000 = 0 \Rightarrow R = -4000 \text{ N}$$

$$\sum M = 0 \Rightarrow -Mc - 4000 * 3 = 0 \Rightarrow Mc = -12000 \text{ N.m}$$

$$0 < x < 3 \Rightarrow Mf^1(x) = Rx - Mc = -4000x + 12000$$

$$3 < x < 5 \Rightarrow Mf^2(x) = Rx - Mc + 4000(x-3) = 0$$

$$0 < x < 3 \Rightarrow \frac{dy}{dx} = \theta^1(x) = \int \frac{Mf(x)}{EI}dx = \frac{1}{EI}\left(-2000x^2 + 12000x + C_1\right)$$

$$0 < x < 3 \Rightarrow y^1(x) = \frac{1}{EI}\left(-\frac{2000}{3}x^3 + 6000x^2 + C_1x + C_2\right)$$

The boundary conditions are:

$$\theta^1(0) = \left(\frac{dy}{dx}\right)_{x=0} = 0 \Rightarrow C_1 = 0 \text{ et } y^1(0) = 0 \Rightarrow C_2 = 0$$

So,

$$0 < x < 3 \Rightarrow y^1(x) = \frac{1}{EI}\left(-\frac{2000}{3}x^3 + 6000x^2\right)$$

$$0 < x < 3 \Rightarrow \theta^1(x) = \frac{1}{EI}\left(-2000x^2 + 12000x\right)$$

For $3 < x < 5$:

Indeed, the bending moment between 3 and 5m is zero, but the deflection in this zone is not zero so the beam remains straight. The slope of this straight line is constant and it is equal to that obtained for the 3m point, i.e. $\theta(3)$:

$$0 < x < 3 \Rightarrow \theta^1(3) = \frac{18000}{EI} = \theta^2(3)$$

$$3 < x < 5 \Rightarrow \theta^2(3) = \frac{18000}{EI} = \frac{y^2(x) - y^1(3)}{(x-3)} = \frac{y^2(x) - (36000/EI)}{(x-3)}$$

$$3 < x < 5 \Rightarrow y^2(x) = \left(\frac{18000}{EI}(x-3)\right) + \frac{36000}{EI} = \frac{18000}{EI}(x-1)$$

So,

$$3 < x < 5 \Rightarrow y_2^2(x) = \frac{18000}{EI}(x-1)$$

$$\left|y_2^2(x)\right|_{Max} = \left|y_2^2(5)\right|_{Max} = \frac{72000}{EI}$$

After superposition, the total maximum deflection will be equal to:

$$\left|y(x)\right|_{Max} = \left|y(5)\right|_{Max} = \left|y_1(5)\right|_{Max} + \left|y_2^2(5)\right|_{Max} = \frac{-93750}{EI} + \frac{72000}{EI} = \frac{-21750}{EI}$$

If E= 200 x 10^9 Pa and I=113 x 10^{-6} m^4, so:

$$\left|y(5)\right|_{Max} = -9{,}6.10^{-4}\ \text{m} \approx -0{,}96\,\text{mm}$$

2.3. Method of moment area

Mathematical method based on the integration of a differential equation can determine the deflection *y(x)* and the slope $\theta(x)$ of a beam at any given point. In this section we will see how with geometric properties of the elastic curve of the bending moment, we can determine the deflection and slope of a beam at a specific point. This latter method is called method of moment area. Figure II-4 gives more detail about this theorem.

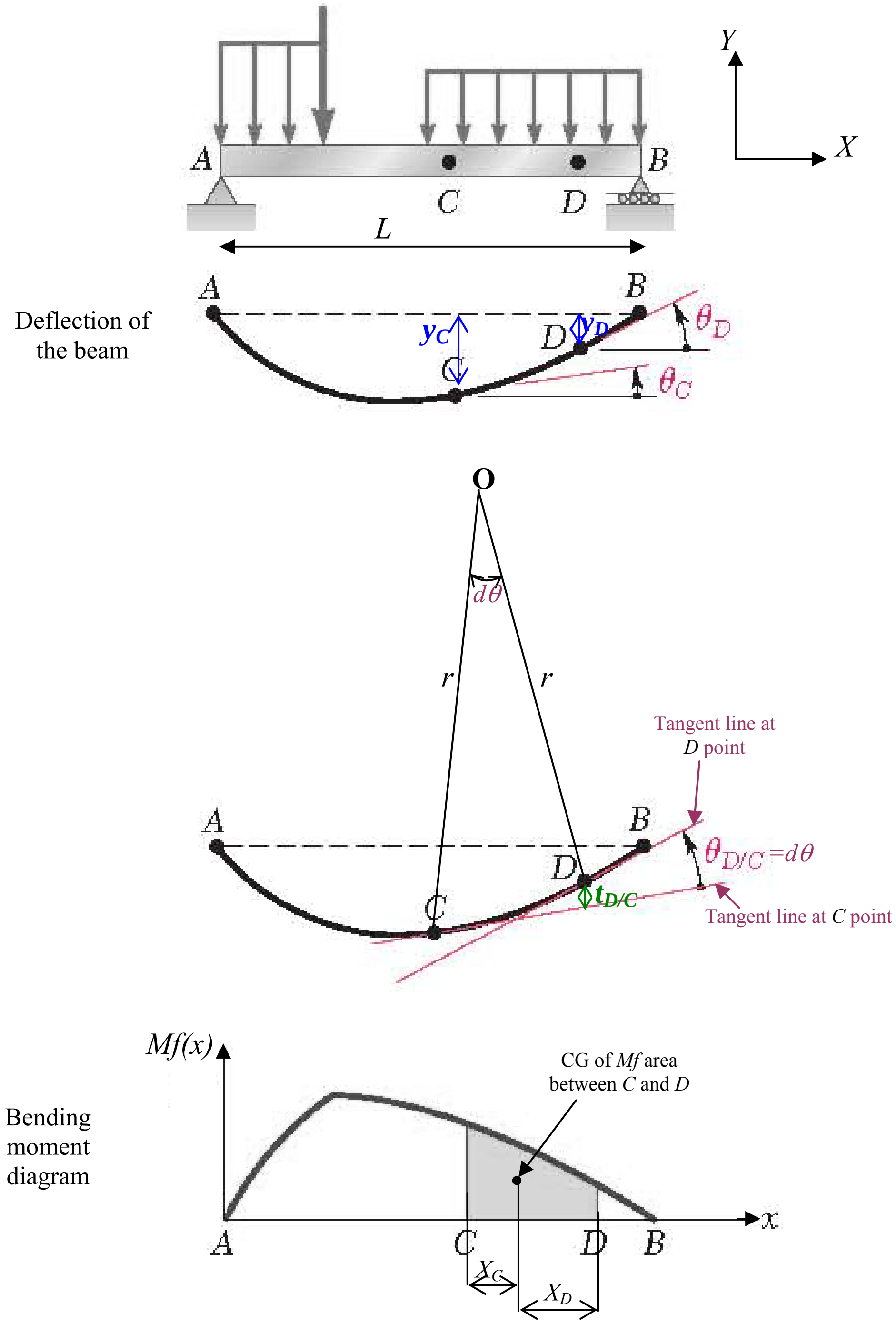


Figure II- 4: Moment area theorem

From the above figure, we like to measure the deflection y_C or y_D corresponding respectively to the C and D points, we know that:

$$\frac{d\theta(x)}{dx} = \frac{Mf(x)}{EI_z} \quad \text{(II-17)}$$

$$\int d\theta(x) = \frac{1}{EI_z}\int Mf(x)dx \quad \text{(II-18)}$$

In the last Mf diagram and for the bending moment area comprised between C and D points Figure II-4, we have:

$$d\theta = \int_{\theta C}^{\theta D} d\theta(x) = \theta_D - \theta_C = \frac{1}{EI_z}\int_{xC}^{xD} Mf(x)dx \quad \text{(II-19)}$$

$$\theta_D - \theta_C = \theta_{D/C} \quad \text{(II-20)}$$

$$\int_{xC}^{xD} Mf(x)dx = A_{DC} \quad \text{(II-21)}$$

$$\Rightarrow \theta_{D/C} = \frac{A_{DC}}{EI_z} \quad \text{(II-22)}$$

A_{CD} represents the area of the bending moment comprised between C and D points (see Figure II-4). From the above figure, we can plot:

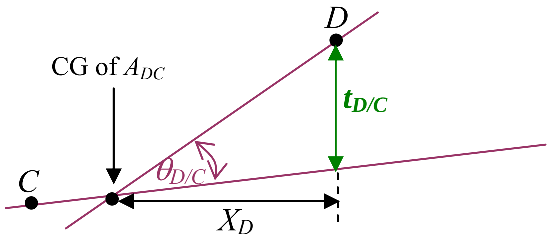


Figure II- 5: Moment area theorem

The intersection point between the two tangents at the C and D points (FigureII-5) presents the gravity center of the area A_{DC} of the bending moment diagram Mf comprised between C and D.

$d\theta = \theta_{D/C} = \theta_D - \theta_C$ is a small angle, so, we can write approximately:

$$tg(d\theta) = tg(\theta_{D/C}) \approx \theta_{D/C} = \frac{t_{D/C}}{X_D} \quad \text{(II-23)}$$

Where X_D is the distance between the D point and the CG point of the A_{DC} moment area (Figure II-4 & Figure II-5). Replacing the equation (II-22) in the equation (II-23), we can find:

$$\theta_{D/C} = \frac{A_{DC}}{EI_z} = \frac{t_{D/C}}{X_D} \quad \text{(II-24)}$$

So, $t_{D/C}$ will be equal to:

$$t_{D/C} = \frac{A_{DC}.X_D}{EI_z} \tag{II-25}$$

Now, we have understanding the calculation procedure of the distance t, we can calculate the deflection y_D at D point (Figure II-6) referring to the areas (A_{BA} and A_{DA}) and with the help of the $t_{B/A}$ and $t_{D/A}$ calculated respectively at the B and D point. We have chosen the B point because the deflection y_B is equal to zero in this point which is the roller support B.

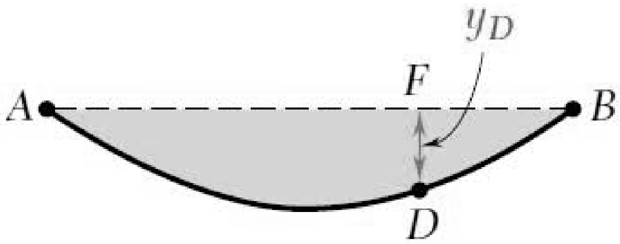


Figure II- 6: The deflection at the *D* point

Firstly, we calculate $t_{B/A}$ using the two tangents at B and A points presented in the below figure (Figure II-7).

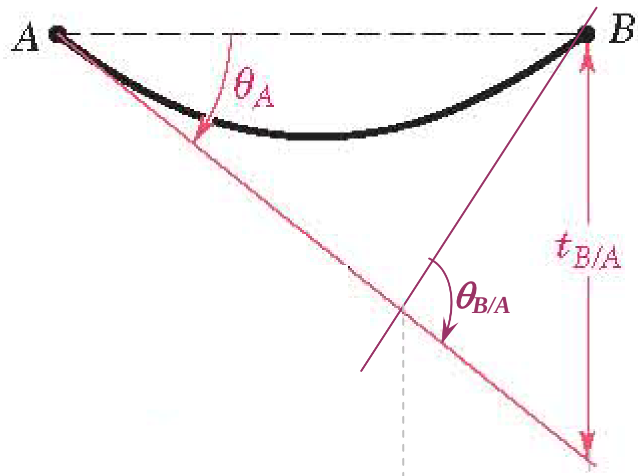


Figure II- 7: The distance $t_{B/A}$

$t_{B/A}$ is equal to:

$$t_{B/A} = \frac{A_{BA}.X_B}{EI_z} \tag{II-26}$$

Where X_B is the distance between the B point and the CG point of the A_{BA} moment area (Figure II-8).

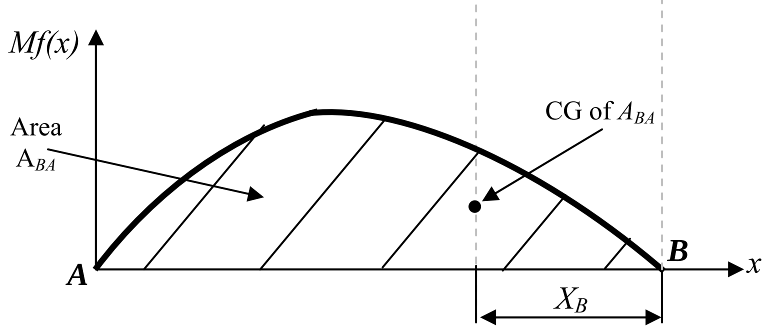


Figure II- 8: X_B and the moment area A_{BA}

Secondly, we calculate $t_{D/A}$ using the two tangents at D and A points presented in the below figure (Figure II-9).

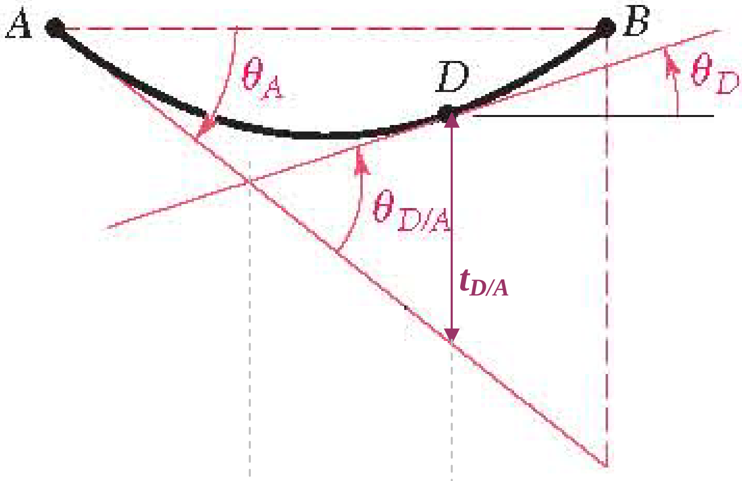


Figure II- 9: The distance $t_{D/A}$

$t_{D/A}$ is equal to:

$$t_{D/A} = \frac{A_{DA}.X_D}{EI_z} \tag{II-27}$$

Where X_D is the distance between the D point and the CG point of the A_{DA} moment area (Figure II-10).

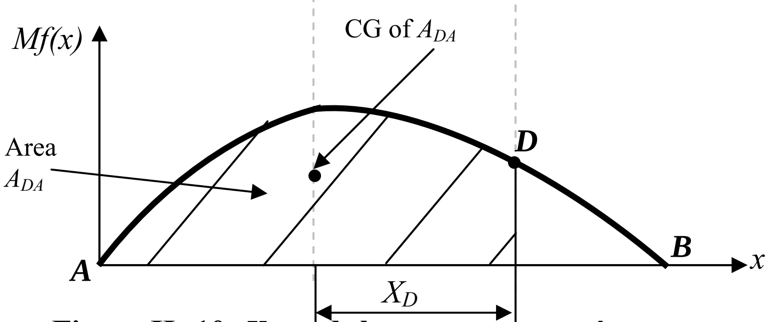


Figure II- 10: X_D and the moment area A_{DA}

Using the following figure to calculate geometrically y_D.

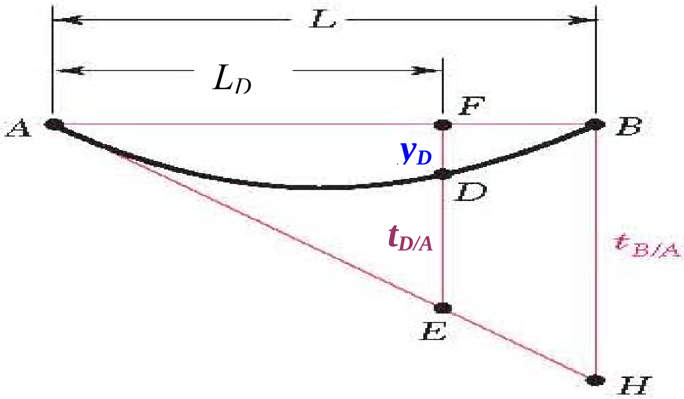


Figure II- 11: Geometrical calculation of the deflection y_D

Knowing L_D, $t_{D/A}$, L and $t_{B/A}$ and with the help of the two triangles AFE and ABH, we can write:

$$EF / AF = HB / AB \Rightarrow EF = AF(HB / AB) = L_D(t_{B/A} / L) \tag{II-28}$$

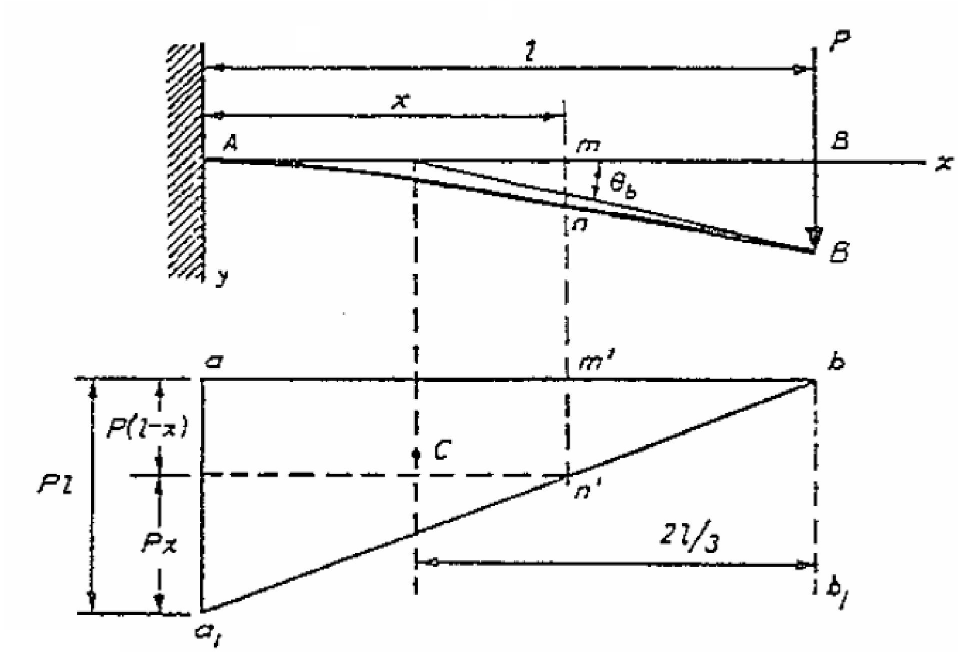
Then y_D will be equal to:

$$y_D = EF - t_{D/A} = L_D(t_{B/A} / L) - t_{D/A} \tag{II-29}$$

Note that we can make the same previous manner to calculate the deflection y_C at C point.

Example:

In this example, we try to calculate the maximum deflection of a cantilever beam (see the below figure) using the two methods studied previously (method of the double-integration and method of the moment area).



Double-integration method:

$$\theta(x)=\int_0^x \frac{Mf(x)}{EI_z}dx=\int_0^x \frac{(P(x-l))}{EI_z}dx=\frac{P}{EI_z}\left(\frac{x^2}{2}-lx\right)$$

$$y(x)=\int_0^x \theta(x)dx=\frac{P}{EI_z}\int_0^x\left(\frac{x^2}{2}-lx\right)dx=\frac{P}{EI_z}\left(\frac{1}{6}x^3-\frac{l}{2}x^2\right)$$

$$y(l)=f_{\max}=-\frac{1}{3}\frac{Pl^3}{EI_z}$$

Moment area method:

Using the two tangents on the *A* and *B* points, the angle $\theta_{B/A}=\theta_B$ because $\theta_A=0$ at x=0; the total bending moment area A_{BA} and the distance X_B which represents the distance between the *B* point and the gravity center of the area A_{BA}, we can find:

$$y(l)=f_{\max}=t_{B/A}=\frac{A_{BA}.X_B}{EI_z}$$

Knowing that: $Mf_{max}=-Pl$; so, $A_{BA}=(Mf_{max}.l)/2=-Pl^2/2$; $X_B=2l/3$, then:

$$y(l)=f_{\max}=t_{B/A}=\frac{A_{BA}.X_B}{EI_z}=-\frac{1}{3}\frac{Pl^3}{EI_z}$$

We found the same f_{max} with two methods.

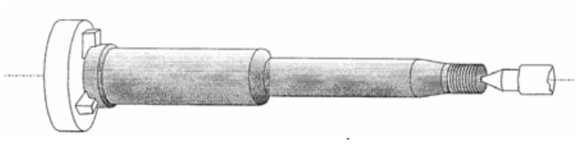
The moment area method has the advantage that is faster than the other methods.

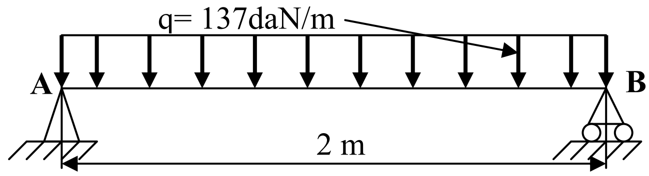
Directed works No. 2 "Deflection of symmetrical beams for pure bending"

Exercise N°1

The beam shown in the figure below is mounted on a lathe machine, the material of the beam is steel with a circular section (diameter d=15 cm), the linear weight of the beam is 137 daN/m, the length of the beam is l = 2m, the Young's modulus of the beam E=200 GPa. The beam to be machined can be modeled as a beam placed on two supports A and B and solicited by its own weight which is uniformly distributed over its entire length. Using the results of the DW N°1 series:

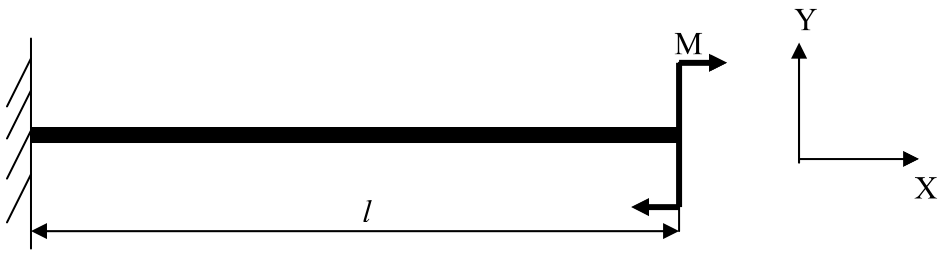
1) Determine the beam deflection equation y(x) by the method of integration of the differential equation.
2) Deduce the maximum deflection of the beam.
3) What will be equal the value of this maximum deflection if we choose a diameter of 10 mm, what can you conclude ?





Exercise N°2

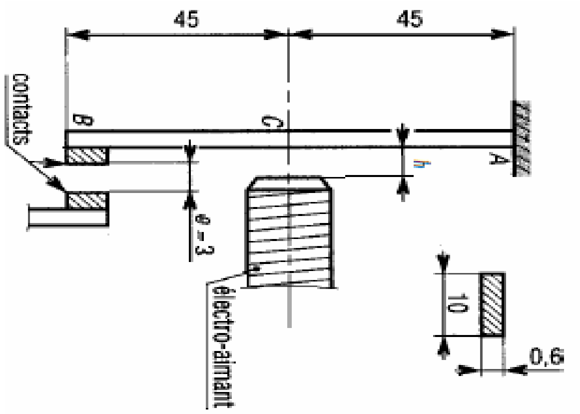
A robot arm exerts a moment M=50000 daN.mm on the free end of an embedded beam (see the figure below), the beam has a diameter d=75 mm, its length *l* is 500 mm.
1) Determine the equation of the deformation of the beam y(x) by the double-integration method and the moment area method.
2) Deduce the maximum deflection of the beam by the two methods knowing that the Young's modulus of the beam E=200 GPa.



Exercise N°3

An electrical contact relay is made from a blade (AB), parallelepiped (90 x 10 x 0.6mm), made of brass and embedded in A. The operation is carried out in C by an electromagnet placed at the distance h from the blade (the electromagnet is at rest). If the gap of the contacts at B is e = 3 mm, determine the necessary force that the electromagnet must exert to establish contact.
From what values of h the contact is possible?

E_{Brass} = 100 GPa.



Chapter III

Energetic methods for elastic systems

Chapter III: Energetic methods for elastic systems

1. Introduction

When a beam is subjected to several loads as bending, torsion, traction or compression, several shearing and normal stresses will be generated at any internal point in this beam (Figure III- 1).

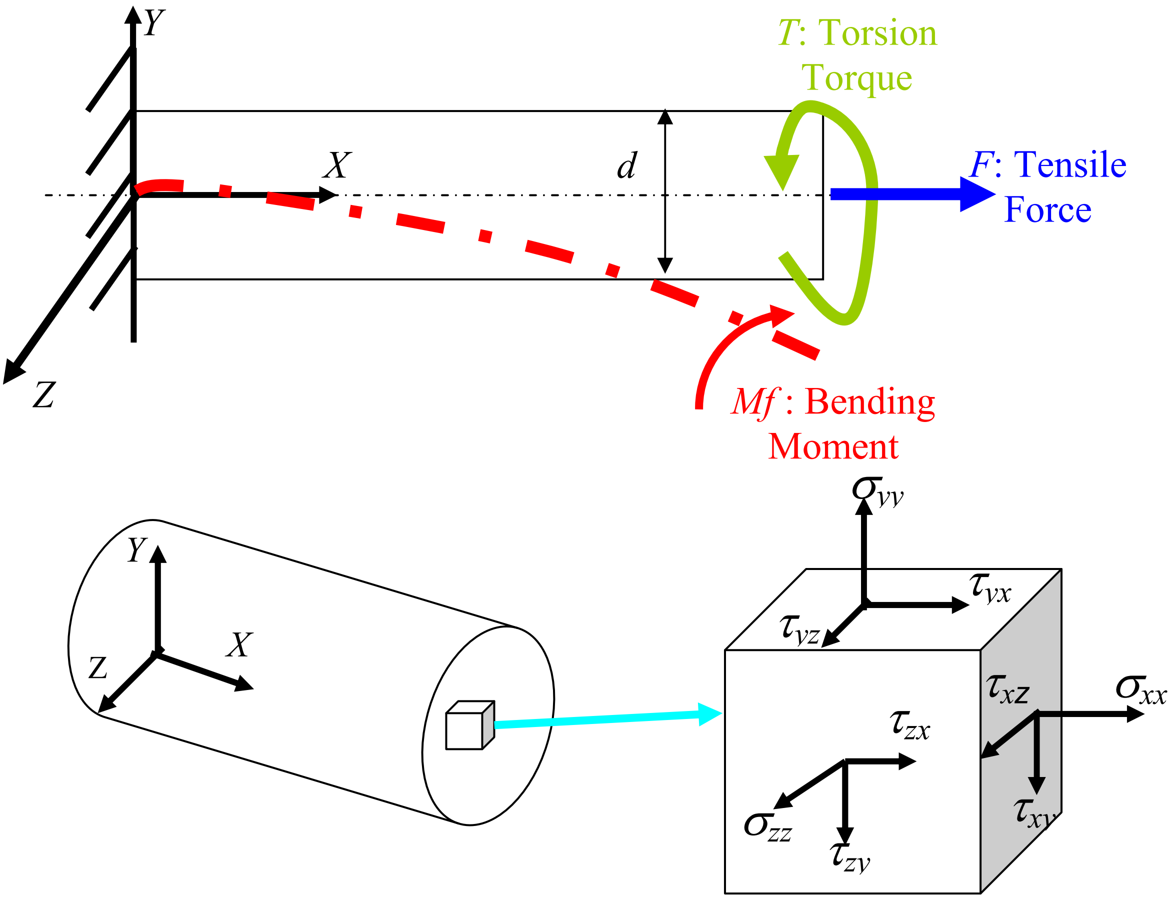


Figure III- 1: Stresses generated in a beam subject to different loads

The stress and the strain symmetrical tensors are equals to:

$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \quad \overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{III-1}$$

In the elastic zone of the beam material, the stresses function strains (**Hooke's law**) are given by:

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz})\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned} \tag{III-2}$$

Where E is the elasticity or the **Young's** modulus of the beam material and ν is the **Poisson's** ratio. (III-2) can be written in the matrix form as:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{pmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{bmatrix}.\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{xz}\\ \varepsilon_{yz}\end{pmatrix} \qquad \text{(III-3)}$$

The elastic behavior law (the **Hooke's law**) of the isotropic material can be simplified to:

$$\overline{\sigma}=\overline{\overline{C}}\,\overline{\varepsilon} \qquad \text{(III-4)}$$

$\overline{\overline{C}}$ is the matrix of the material elastic constants.

From the equation (III-2), we can express the strains function stresses, so, the strain-stress relations are given by:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx}-\nu\left(\sigma_{yy}+\sigma_{zz}\right)\right]\\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy}-\nu\left(\sigma_{xx}+\sigma_{zz}\right)\right]\\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{xx}+\sigma_{yy}\right)\right]\\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}\\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz}\\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned} \qquad \text{(III-5)}$$

For the isotropic linear elastic material, the elastic shear modulus is equal to:

$$\mu=\frac{E}{2(1+\nu)} \qquad \text{(III-6)}$$

Equation (III-5) can be written:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx}-\nu\left(\sigma_{yy}+\sigma_{zz}\right)\right]\\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy}-\nu\left(\sigma_{xx}+\sigma_{zz}\right)\right]\\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{xx}+\sigma_{yy}\right)\right]\\
\varepsilon_{xy} &= \frac{1}{2\mu}\sigma_{xy}\\
\varepsilon_{xz} &= \frac{1}{2\mu}\sigma_{xz}\\
\varepsilon_{yz} &= \frac{1}{2\mu}\sigma_{yz}
\end{aligned} \qquad \text{(III-7)}$$

Normal stresses produce volume changes and shear stresses produce a change in shape (distortion). The relation $\sigma_{xy} = \mu(2\varepsilon_{xy})$ can be written $\tau_{xy} = \mu\gamma_{xy}$, where τ_{xy} is the shearing stress, ε_{xy} is the shear strain or the shear deformation and γ_{xy} is the angular strain. So, (III-7) relations can be expressed as:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\gamma_{xy} &= \frac{\tau_{xy}}{\mu} \\
\gamma_{xz} &= \frac{\tau_{xz}}{\mu} \\
\gamma_{yz} &= \frac{\tau_{yz}}{\mu}
\end{aligned}
\qquad \text{(III-8)}
$$

2. General relation of the elastic strain energy

2.1. Work of axial loading

The below figure show the axial strain induced by the application of an axial force in a rod; the objective is to show how we calculate the axial elastic strain work or energy of this rod.

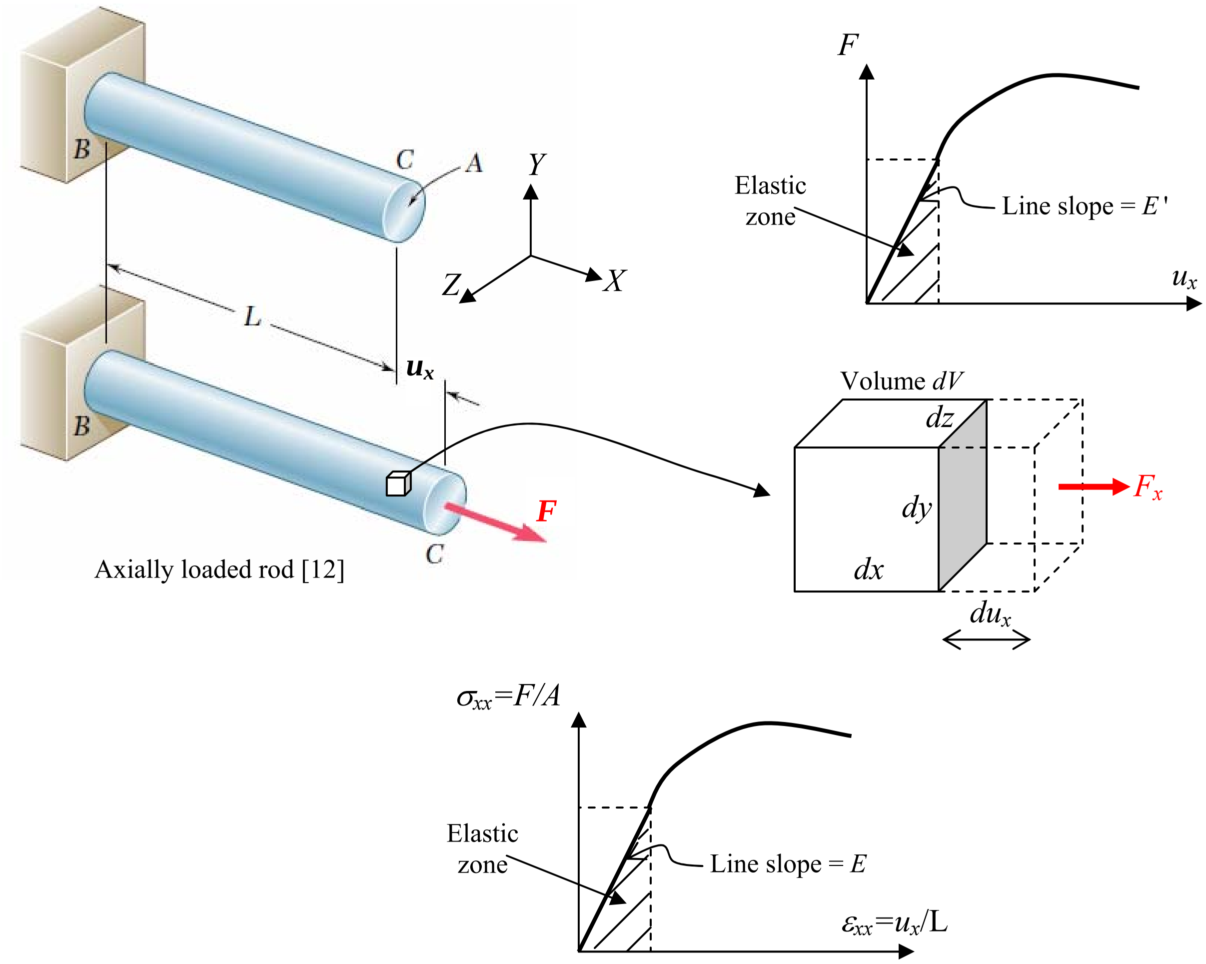


Figure III- 2: Generated work and stress versus strain for axially loaded rod

In the above figure, after the application of the force *F*, the rod deforms axially along the *X* axis by a value of u_x. The global translation work in Joule or N.m is equal to:

$$W = F.u_x = F_x.du_x \tag{III-9}$$

If we have a torque load *T* applied around the *X* axis (Figure III- 2) which generates an angle θ_x, then the rotation work is equal to:

$$W = T.\theta_x \tag{III-10}$$

When we differentiate the (III-9), we obtain:

$$dW = F.du_x \tag{III-11}$$

In the elastic part $F = E'.u_x$ where E' is the rod stiffness in N/m.

(III-11) becomes:

$$\begin{aligned} & dW = E'u_x.du_x \\ & \Rightarrow W = \int E'u_x.du_x \\ & \Rightarrow W = \frac{1}{2}E'u_x^2 = \frac{1}{2}E'u_x u_x \\ & \Rightarrow W = \frac{1}{2}F.u_x \end{aligned} \tag{III-12}$$

Note that if *E* (**Young's** modulus in N/m^2) is constant and if the rod cross-section area decreases from *A* to *dydz*, therefore, the applied force also decreases from *F* to F_i with the same proportionality rate of the decrease of this area; but the stress σ_{xx} remains exactly the same. Moreover, the extension decreases from u_x to du_x with the same proportionality rate that exists between *L* and *dx*, but the deformation ε_{xx} remains constant. In addition, for an infinitesimal volume *dV* (Figure III- 2), the work of the elementary force F_x will be equal:

$$dW = \frac{1}{2}F_x.du_x \tag{III-13}$$

We know that:

$$\sigma_{xx} = \frac{F}{A} = \frac{F_x}{dydz} \Rightarrow F_x = \sigma_{xx}.dydz \tag{III-14}$$

Also:

$$\varepsilon_{xx} = \frac{u_x}{L} = \frac{du_x}{dx} \Rightarrow du_x = \varepsilon_{xx}dx \tag{III-15}$$

Replacing equations (III-14) and (III-15) in the equation (III-13), we can find:

$$\begin{aligned} & dW = \frac{1}{2}\sigma_{xx}.dydz.\varepsilon_{xx}dx \\ & \Rightarrow dW = \frac{1}{2}\sigma_{xx}.\varepsilon_{xx}dxdydz \\ & \Rightarrow dW = \frac{1}{2}\sigma_{xx}.\varepsilon_{xx}dV \end{aligned} \tag{III-16}$$

We know that for a tensile loading, the **Hooke's** law will be equal in the elastic zone:

$$\sigma_{xx} = E\varepsilon_{xx} \quad \text{(III-17)}$$

Where E is the elastic tensile modulus, then the equation (III-16) can be written as:

$$dW = \frac{1}{2}E.\varepsilon_{xx}^{2}dV \quad \text{(III-18)}$$

2.2. Work of shear loading

In the following figure, the shear force T_y generates the shearing stress τ_{xy} and produces a shape change in the below infinitesimal volume.

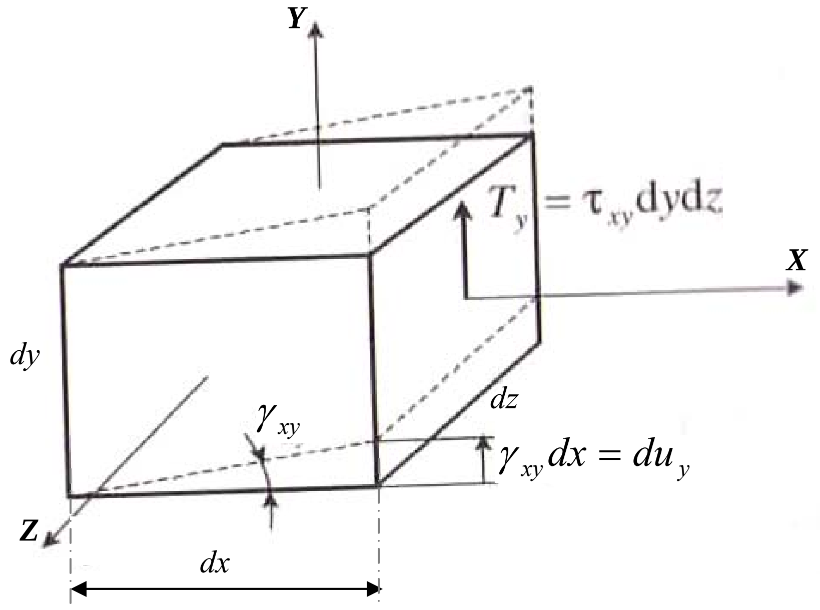


Figure III- 3: Generated work for shear loading

As we have demonstrated latter, the work of the elementary shear force T_y is written in the form of:

$$dW = \frac{1}{2}T_y.du_y \quad \text{(III-19)}$$

We know that:

$$\tau_{xy} = \frac{T_y}{dydz} \Rightarrow T_y = \tau_{xy}.dydz \quad \text{(III-20)}$$

Also:

$$2\varepsilon_{xy} = \gamma_{xy} = \frac{du_y}{dx} \Rightarrow du_y = 2\varepsilon_{xy}dx = \gamma_{xy}dx \quad \text{(III-21)}$$

Replacing equations (III-20) and (III-21) in the equation (III-19), we can find:

$$\begin{aligned} & dW = \frac{1}{2}\tau_{xy}.dydz.\gamma_{xy}dx \\ & \Rightarrow dW = \frac{1}{2}\tau_{xy}.\gamma_{xy}dxdydz \\ & \Rightarrow dW = \frac{1}{2}\tau_{xy}.\gamma_{xy}dV \\ & \Rightarrow dW = \tau_{xy}.\varepsilon_{xy}dV \end{aligned} \quad \text{(III-22)}$$

We know that for a tensile loading, the **Hooke's** law will be equal in the elastic zone:

$$\tau_{xy} = \mu\gamma_{xy} = 2\mu\varepsilon_{xy} \quad \text{(III-23)}$$

Where μ is the elastic shear modulus, then the equation (III-22) can be written as:

$$dW = \frac{1}{2}\mu.\gamma_{xy}^2 dV \quad \text{(III-24)}$$

We can also write (III-24) as:

$$dW = 2\mu.\varepsilon_{xy}^2 dV \quad \text{(III-25)}$$

2.3. Strain energy

The total elementary strain energy *dU* in Joule is equal to the sum of all the elementary axial and shear works which are generated in the three directions (*X*, *Y* and *Z*). *dU* will be equal to:

$$dU = \sum_{i=1}^{6} dW_i \quad \text{(III-26)}$$

We can write the previous relation as:

$$dU = dW_{xx} + dW_{yy} + dW_{zz} + dW_{xy} + dW_{xz} + dW_{yz} \quad \text{(III-27)}$$

dW_{xx}, dW_{yy} and dW_{zz} represent the elementary works respectively along the *X*, *Y* and *Z* axes.

dW_{xy}, dW_{xz} and dW_{yz} represent the elementary works in the three directions *X*, *Y* and *Z* axes.

$$dU = \frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz} + \tau_{xy}.\gamma_{xy} + \tau_{xz}.\gamma_{xz} + \tau_{yz}.\gamma_{yz}\right)dV \quad \text{(III-28)}$$

$$dU = \frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz} + 2\tau_{xy}.\varepsilon_{xy} + 2\tau_{xz}.\varepsilon_{xz} + 2\tau_{yz}.\varepsilon_{yz}\right)dV \quad \text{(III-29)}$$

$$dU = \left[\frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz}\right) + \left(\tau_{xy}.\varepsilon_{xy} + \tau_{xz}.\varepsilon_{xz} + \tau_{yz}.\varepsilon_{yz}\right)\right]dV \quad \text{(III-30)}$$

The total elastic strain energy *U* for a volume *V* can be written as:

$$U = \int\left[\frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz}\right) + \left(\tau_{xy}.\varepsilon_{xy} + \tau_{xz}.\varepsilon_{xz} + \tau_{yz}.\varepsilon_{yz}\right)\right]dV \quad \text{(III-31)}$$

The strain-energy density *u* in Joule/m^3 or N/m^2 is defined as the total elastic strain energy *U* in Joule divided by the structure volume in m^3:

$$u = \frac{dU}{DV} \quad \text{(III-32)}$$

Using the two equations (III-28) and (III-30), we find the strain-energy density *u*:

$$u = \frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz} + \tau_{xy}.\gamma_{xy} + \tau_{xz}.\gamma_{xz} + \tau_{yz}.\gamma_{yz}\right) \quad \text{(III-33)}$$

$$u = \frac{1}{2}\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \sigma_{zz}.\varepsilon_{zz}\right) + \left(\tau_{xy}.\varepsilon_{xy} + \tau_{xz}.\varepsilon_{xz} + \tau_{yz}.\varepsilon_{yz}\right) \quad \text{(III-34)}$$

The unit of u is N/m^2. Using the equation (III-8), the strain-energy density u can be expressed only in function of stresses as:

$$u = \frac{1}{2E}\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right) - \frac{\nu}{E}\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right) + \frac{1}{2\mu}\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right) \quad \text{(III-35)}$$

Using the equation (III-2), the strain-energy density u can be expressed only in function of strains as:

$$\begin{aligned} \boldsymbol{u} &= \frac{\mu}{1-2\nu}\left[(1-\nu)\left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2\right) + 2\nu\left(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}\right)\right] + 2\mu\left(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2\right) \\ &= \frac{\nu\mu}{1-2\nu}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)^2 + \mu\left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2\right) + 2\mu\left(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2\right) \end{aligned} \quad \text{(III-36)}$$

Note that in a structural element or machine part with a nonuniform stress distribution, the determination of the strain-energy density u is necessary.

3. Elastic strain energy in traction or compression

The below figure show the behavior law for a bar material subjected to a tensile test. The elastic zone is limited by the yield stress σ_Y and the yield strain ε_Y.

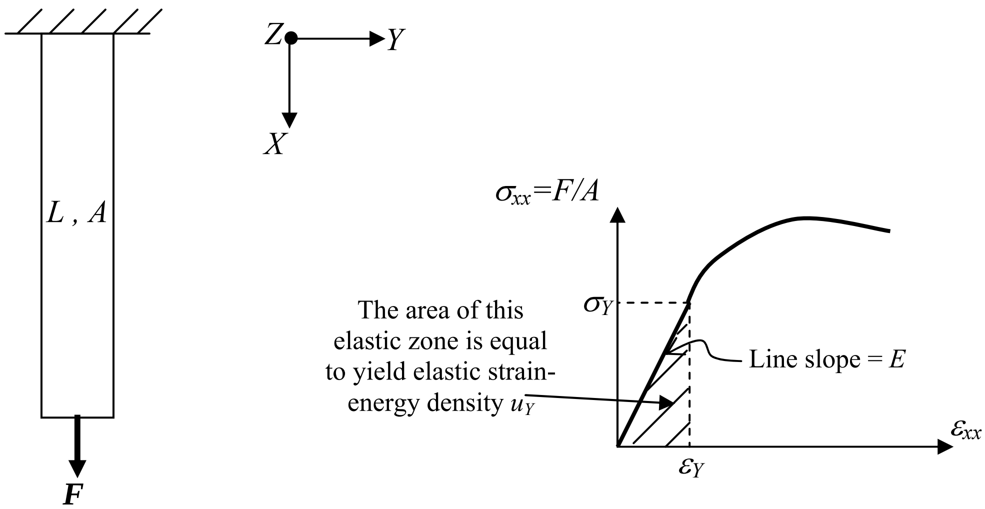


Figure III- 4: Tensile test and elastic zone

Using the equations (III-1) and (III-5), the symmetrical tensors of the stress and the strain are equals in the case of a tensile test to:

$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} = -\nu\varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\nu\varepsilon_{xx} \end{bmatrix} \quad \text{(III-37)}$$

Replacing the components of the previous tensors in the equation (III-3) of the strain-energy density u, we find:

$$u = \frac{1}{2}\sigma_{xx}.\varepsilon_{xx} \quad \text{(III-38)}$$

The elastic strain-energy density u represents the area under the elastic straight-line of the stress-strain diagram (Figure III- 4) corresponding to the values of σ_{xx} and ε_{xx} .

(III-38) can be also written as:

$$U = \frac{1}{2}\sigma_{xx}.\varepsilon_{xx}.V = \frac{1}{2}\frac{\sigma_{xx}^2}{E}V \qquad \text{(III-39)}$$

U is the elastic strain energy in Joule and V is the volume of the bar.

The maximum elastic strain-energy density signifies that a material can store or absorb energy without yielding or without undergoing plastic deformation is given by the following relationship:

$$u_Y = \frac{1}{2}\sigma_Y.\varepsilon_Y \qquad \text{(III-40)}$$

Where σ_Y and ε_Y represent respectively the material yield stress and yield strain, u_Y is called also the yield elastic strain-energy density and is equal to the total area of the elastic zone (Figure III- 4). The latter is also known under the name the modulus of resilience. The capacity of a structure to withstand an impact load without being permanently deformed clearly depends upon the resilience of the material used.

(III-40) can be also written as:

$$U_Y = \frac{1}{2}\sigma_Y.\varepsilon_Y.V = \frac{1}{2}\frac{\sigma_Y^2}{E}V \qquad \text{(III-41)}$$

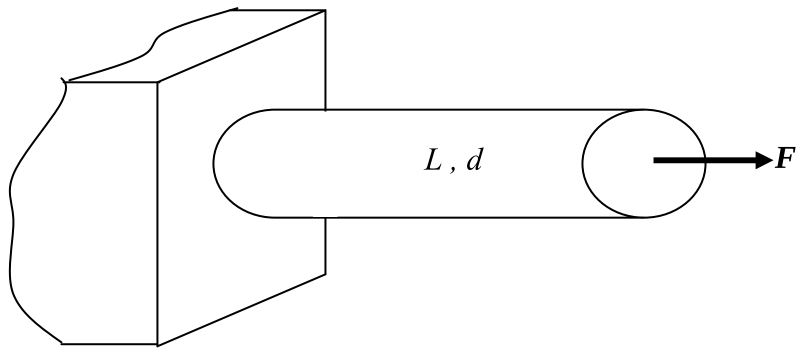
U_Y is the yield strain energy; the strain energy verification criterion allows us to know the energy U which can be applied without having a risk to deform plastically a mechanical part. The strain energy criterion is given by:

$$Fs \times U < U_Y \qquad \text{(III-42)}$$

Where, Fs is the safety factor used to avoid permanent deformation. We can see in the (III-39) equation, that there is no linearity between U and σ_Y . For this reason; the safety factor Fs is applied to the strain-energy U and not to the stress.

Example 1:

In this example we like to calculate the required maximum yield stress σ_Y that a rod material should have in order to does not undergo a permanent deformation and that after the application of an axial force of magnitude F equal to 33.74 kN, the elastic modulus of the material is E=200 GPa, the diameter of the cross-section area of this rod is d=20mm and its length L=1.5m, the Fs factor of safety is equal to 5.



Let's calculate firstly, the area of the rod cross-section:

$$A = \pi \frac{d^2}{4} = \pi \frac{0{,}02^2}{4} = 3.14 \times 10^{-4}\,\text{m}^2$$

The stress σ is equal to:

$$\sigma = \frac{F}{A} = \frac{33740}{3.14.10^{-4}} = 1.074 \times 10^8\,\text{N/m}^2 \text{ ou Pa} = 107.4\,\text{MPa}$$

The strain energy generated by the force F is equal to:

$$U = \frac{1}{2}\frac{\sigma^2}{E}V = \frac{1}{2}\frac{\sigma^2}{E}A.L = 0.5 \times \frac{1.074.10^8}{200 \times 10^9} \times 3.14 \times 10^{-4} \times 1.5 = 13.6\,\text{N.m} = 13.6\,\text{Joule}$$

As was indicated in the (III-42) equation, to have more safety, the value of the strain-energy U must be majored by the safety factor Fs:

$$Fs \times U = 5 \times 13.6 = 68\,\text{N.m}$$

$$Fs \times U < \frac{1}{2}\frac{\sigma_Y^2}{E}V \text{ or } Fs \times U < \frac{1}{2}\frac{\sigma_Y^2}{E}A \times L$$

Then, σ_Y must be superior to:

$$\sigma_Y > \sqrt{\frac{2E \times Fs \times U}{A \times L}} \Rightarrow \sigma_Y > \sqrt{\frac{2 \times 200 \times 10^9 \times 5 \times 13.6}{3.14 \times 10^{-4} \times 1.5}} \Rightarrow \sigma_Y > 2.4 \times 10^8\,\text{Pa} \Rightarrow \sigma_Y > 240\,\text{MPa}$$

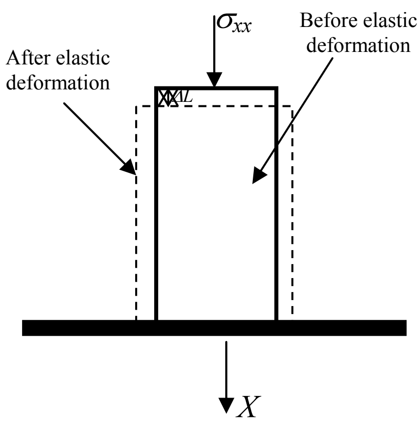
So, we must use a material with a yield stress σ_Y higher than 240 MPa as not to have a permanent deformation of the rod material after the application of the force F of 33.74 kN.

Example 2:

In this example we like to calculate the strain energy for a rod when it will be submitted to stress σ_{xx} equal to 300 MPa, (E=200 GPa, A=90 square mm and L=3m):

$$U = \frac{1}{2}\sigma_{xx}.\varepsilon_{xx}.V = \frac{1}{2}\frac{\sigma_{xx}^2}{E} \times V = \frac{1}{2}\frac{\sigma_{xx}^2}{E} \times A \times L$$

$$U = \frac{1}{2}\frac{300^2}{200000} \times 90 \times 3000 = 60750\,\text{N.mm} = 60.75\,\text{Joule}$$



After the application of this stress, the value of the compression ΔL is:

$$\Delta L = \frac{\sigma_{xx} L}{E}$$

$$\Delta L = \frac{300 \times 3000}{200000} = 4.5\,\text{mm}$$

3.1 Toughness modulus and resilience modulus

a) Resilience modulus

The modulus of resilience is equal to the total area of the elastic zone (Figure III- 5) which represents the elastic strain-energy density u_Y, it represents the capacity of a structure to withstand an impact load without being permanently deformed. Resilience modulus or u_Y gives also an index for the ability of materials to absorb or store energy without permanent deformation. In the below figure, it is clear that the material B is more resilient and resist plus to the plastic deformation (more storage and absorption of energy) than the material A.

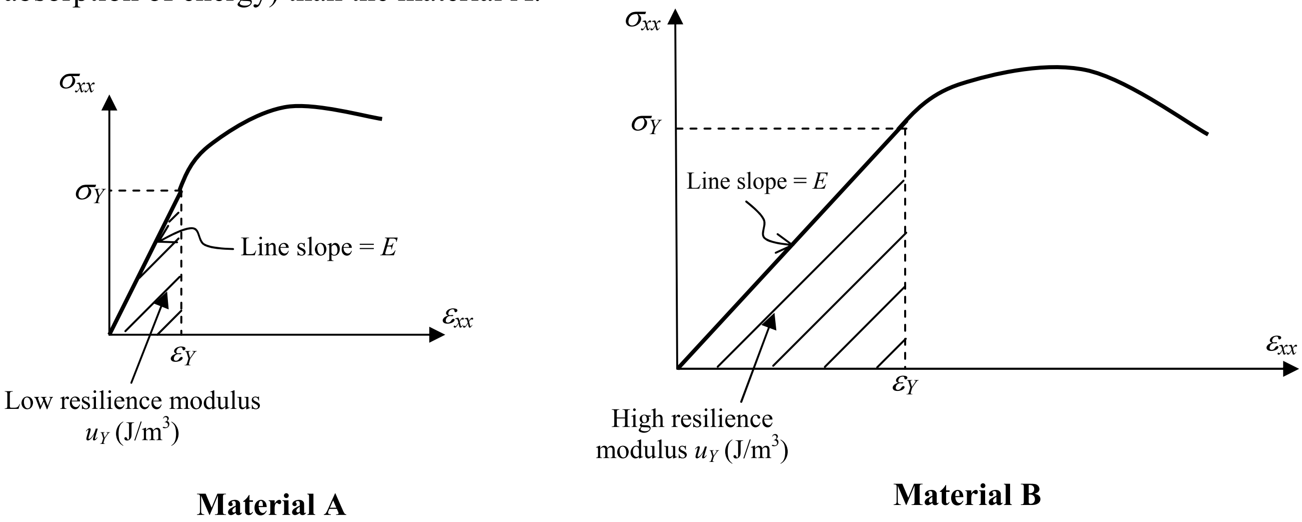


Figure III- 5: Modulus of resilience

b) Toughness modulus

The toughness is equal to the area under the entire stress-strain diagram (

Figure III- 6), it is defined as the ability of a material to absorb energy up to fracture or as the energy per unit volume required to cause the material to rupture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength σ_U and that the capacity of a structure to withstand an impact load depends upon the toughness of the material used. As before, material B is more resistant to impacts than material A.

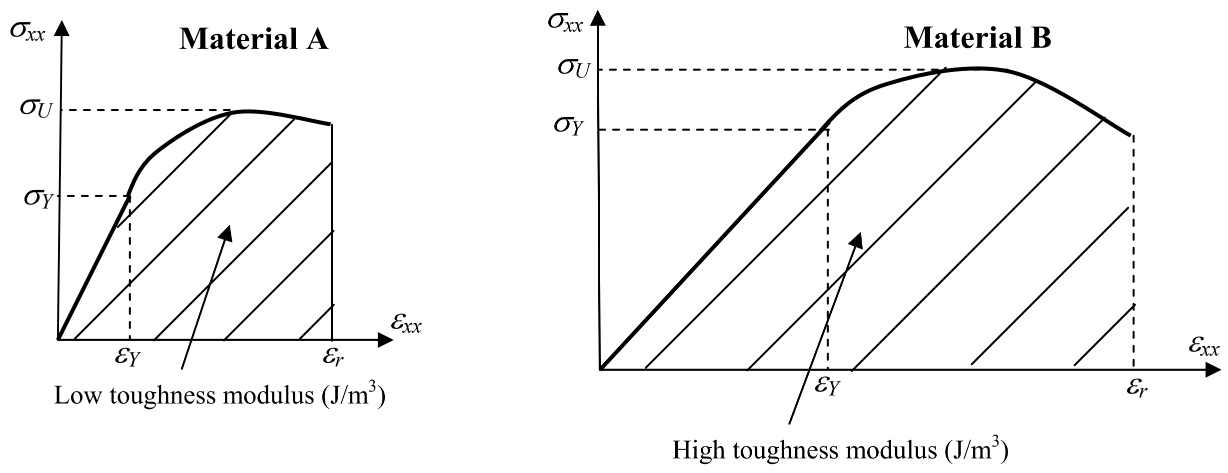


Figure III- 6: Modulus of toughness

3.2 Castigliano theorem to calculate displacement

The Castigliano theorem allows calculating the displacements u_{xi}, u_{yi} or u_{zi} in a point i respectively along the three axes X, Y or Z of the structure. It is defined as the derivative of the total elastic strain energy with respect to the force F_i applied in this point i:

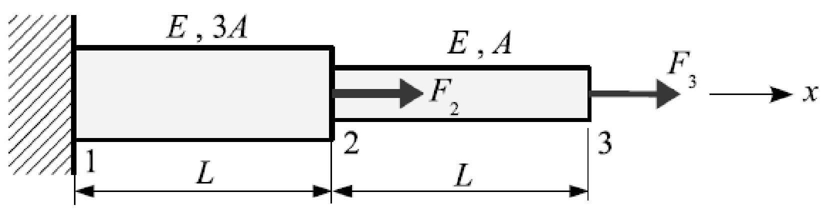
$$u_{xi} = \frac{\partial U^{Total}}{\partial F_i} \qquad \text{(III-43)}$$

Referring to the figure (Figure III- 4) and knowing that $\sigma = E\varepsilon$, the elastic strain energy in the case of traction or compression loading is equal after indices simplification to:

$$U^{Tensile} = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V \frac{\sigma^2}{E} dV = \frac{1}{2}\int \frac{F^2}{EAA} A dx = \int \frac{F^2}{2EA} dx \qquad \text{(III-44)}$$

Example 3:

The bar shown in the figure is embedded at 1. Let E be the Young's modulus of the material. The area of the cross section is $3A$ between the points 1 and 2 and A between the points 2 and 3. This bar carries in 2 a force with components (F_2, 0, 0) and in 3 a force with components (F_3, 0, 0) [15].



Let calculate:

1. The expression of the normal force *N(x)*:

$$\sum F_{/x} = 0 \Rightarrow Rx - F_2 - F_3 = 0 \Rightarrow Rx = F_2 + F_3$$
$$0 < x < L \Rightarrow N(x) = N_{12} = Rx = F_2 + F_3$$
$$L < x < 2L \Rightarrow N(x) = N_{23} = Rx - F_2 = F_2 + F_3 - F_2 = F_3$$

Then:

$N_{12} = F_2 + F_3$ and $N_{23} = F_3$

2. The total elastic strain energy U^{Total}:

$$U^{Total} = \int_0^L \frac{N_{12}^2}{2E3A} dx + \int_L^{2L} \frac{N_{23}^2}{2EA} dx = \frac{L}{2EA}\left[\frac{1}{3}(F_2 + F_3)^2 + F_3^2\right]$$

3. The displacements u_{x2} and u_{x3} respectively in the points 2 and 3:

$$u_{x2} = \frac{\partial U^{total}}{\partial F_2} = \frac{L}{3EA}(F_2 + F_3) \;;\; u_{x3} = \frac{\partial U^{total}}{\partial F_3} = \frac{L}{3EA}(F_2 + 4F_3)$$

4. The flexibility and the stiffness matrices of this bar:

Flexibility matrix $[C]$:

$$\begin{pmatrix} u_{x2} \\ u_{x3} \end{pmatrix} = \frac{L}{3EA}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{pmatrix} F_2 \\ F_3 \end{pmatrix} \Rightarrow [C] = \frac{L}{3EA}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

Rigidity matrix $[K]$:

$$\begin{pmatrix} F_2 \\ F_3 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_{x2} \\ u_{x3} \end{pmatrix} \Rightarrow [K] = [C]^{-1} = \frac{EA}{L}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

4. Elastic strain energy in bending

In the case of a bending loading that will held in the *XY* plane and around the *Z* axis, we have the bending strain energy due to the normal tensile and compressive stresses generated by the bending moments Mf_z and the shear strain energy induced by the shear forces T_y. The shear strain energy is neglected compared to the bending strain energy. The bending and shear elastic strain energy is equal to:

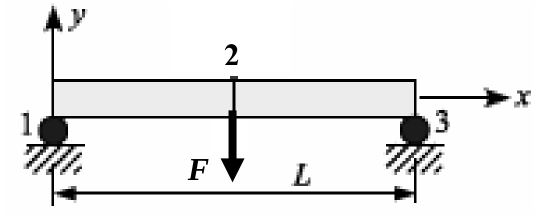
$$U^{Bending} = \frac{1}{2}\int_V \sigma\varepsilon dV + \frac{1}{2}\int_V \tau\gamma dV = \frac{1}{2}\iiint \frac{\sigma^2}{E}dV + \frac{1}{2}\iiint \frac{\tau^2}{\mu}dV \qquad \text{(III-45)}$$

$\sigma = \frac{Mf_z.y}{I_z}$; $\tau = \frac{T_y}{A}$; I_z and A are the inertia moment and the area of the beam cross section.

$$\begin{aligned} U^{Bending} &= \frac{1}{2}\iiint \frac{Mf_z^2 y^2}{EI_z^2}dsdx + \frac{1}{2}\int \frac{T_y^2}{\mu A^2}Adx \\ &= \frac{1}{2}\int \frac{Mf_z^2}{EI_z^2}dx\iint y^2 ds + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \\ &= \frac{1}{2}\int \frac{Mf_z^2 I_z}{EI_z^2}dx + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \\ &= \frac{1}{2}\int \frac{Mf_z^2}{EI_z}dx + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \end{aligned} \qquad \text{(III-46)}$$

Example 1:

The following beam of length L and constant section (quadratic moment: I_z) is supported at 1 and 3 on a simple support. The beam is made with steel having a **Young's** modulus E. It carries in its center (middle point 2) a force with components (0, F, 0).



We neglect the influence of the shear force: Bernoulli model. Let calculate:

1. The expression of the bending moment $Mf_z(x)$:

$$0 < x < \frac{L}{2} \Rightarrow Mf_1(x) = Mf_{12} = \frac{F}{2}x$$

$$\frac{L}{2} < x < L \Rightarrow Mf_2(x) = Mf_{23} = \frac{F}{2}(L-x)$$

2. The total elastic strain energy U^{Total} :

$$U^{Total} = \frac{1}{2EI_z}\left(\int_0^{L/2} Mf_{12}^2 dx + \int_{L/2}^{L} Mf_{23}^2 dx\right) = \frac{F^2}{8EI_z}\left(\int_0^{L/2} x^2 dx + \int_{L/2}^{L} (L-x)^2 dx\right)$$

$$= \frac{F^2}{8EI_z}\left(\frac{1}{3}x^3\Big|_0^{L/2} - \frac{1}{3}(L-x)^3\Big|_{L/2}^{L}\right)$$

$$U^{Total} = \frac{F^2L^3}{96EI_z}$$

3. The deflection in point 2:

$$u_{y2} = \frac{\partial U^{total}}{\partial F} = \frac{FL^3}{48EI_z}$$

$\Rightarrow F = \frac{48EI_z}{L^3} \times u_{y2} \Rightarrow$ The bending rigidity is equal to $\frac{48EI_z}{L^3}$; it depends proportionally to E and I_z and inversely to L.

4. Comparison of the deflection of the point 2 calculated with Castigliano theorem with the deflection calculated by the double-integration method:

Using the double-integration method seen in the Chapter II, the deflection *y(x)* for this example is equal to:

$$y_1(x) = \frac{1}{4EI_z}\left(\frac{F}{3}x^3 - \frac{FL^2}{4}x\right)$$

Now, the maximum deflection f_{max} is located in the center of the beam, and then f_{max} is equal to:

$$f_{\max} = y_1(L/2) = -\frac{FL^3}{48EI_z} \Rightarrow u_{y2} = |f_{\max}|$$

We find the same expression obtained by the Castigliano theorem

5. Elastic strain energy in torsion

5.1 Torsion rigidity and shearing stress induced by torsion

The twist angle θ must be less than 1° over a shaft length equal to 20 times the shaft diameter. The example presented in the below figure shows a shaft of diameter *d*, radius *R* and length *L* subjected to a torque M_t.

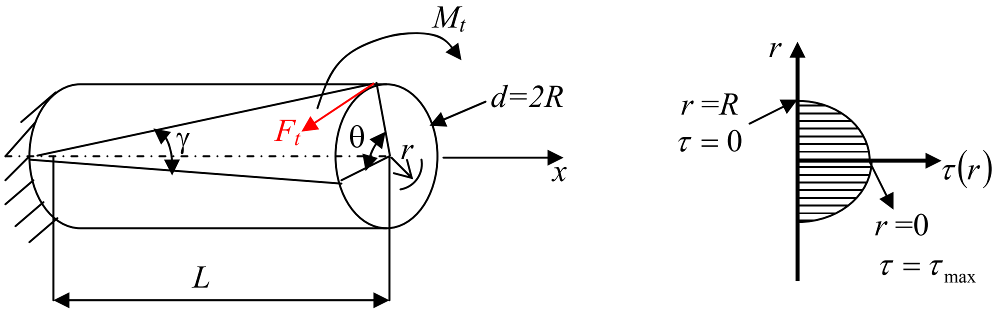


Figure III- 7: Torsion loading and its generated shearing stress

We know that:

$$dM_t = \tau(r) \times r \times ds \tag{III-47}$$

$\tau(r)$ is the shear stress at a point on the cross-section area A which is located at a radius r from the center of this area A. ds is a small area of the total area A. Thus, $\tau(r)$ is equal to:

$$\tau(r) = \mu \times \gamma(r) = \mu \times r \times \frac{d\theta}{dx} \quad \text{with } \gamma(r) = \frac{rd\theta}{dx} = \frac{R\theta}{L} \tag{III-48}$$

$$\text{(III-47)} \Rightarrow \int dM_t = \int \mu \times r \times \frac{d\theta}{dx} \times r \times ds$$

$$\Rightarrow M_t = \mu \times \frac{d\theta}{dx} \times \int r^2 ds = \mu \times \frac{d\theta}{dx} \times I_p \tag{III-49}$$

$$\Rightarrow \int_0^L M_t dx = \int_0^\theta \mu \times I_p \times d\theta$$

After integration we can write:

$$M_t = \frac{\mu \times I_p}{L}\theta = k_t.\theta = F_t \times R \tag{III-50}$$

k_t is the torsional rigidity in N.m, it is equal to:

$$K_{tors} = \frac{\mu \times I_P}{L} \tag{III-51}$$

The elastic shear modulus μ is equal to:

$$\mu = \frac{E}{2(\nu + 1)} \tag{III-52}$$

The polar inertia moment I_p is calculated by the following formula:

$$I_p = \int r^2 ds = \frac{\pi d^4}{32} = \frac{\pi R^4}{2} \tag{III-53}$$

The twist angle θ in radians is calculated like this:

$$\theta = \frac{M_t.L}{\mu \times I_p} \tag{III-54}$$

5.2 Torsion strain energy

We know that from equation (III-48):

$$\tau(r) = \mu \times r \times \frac{d\theta}{dx} \tag{III-55}$$

And from equation (III-49), we have:

$$\frac{d\theta}{dx} = \frac{M_t}{\mu \times I_p} \tag{III-56}$$

Substituting (III-56) into (III-55), we obtain:

$$\tau(r) = \mu \times r \times \frac{M_t}{\mu \times I_p}$$
$$\Rightarrow \tau(r) = \frac{M_t \times r}{I_p} \tag{III-57}$$

$$\Rightarrow \gamma(r) = \frac{\tau(r)}{\mu} = \frac{M_t \times r}{\mu \times I_p} \tag{III-58}$$

The torsional strain energy is given by:

$$U^{Torsion} = \frac{1}{2}\int_V \tau\gamma dV = \frac{1}{2}\iiint \frac{\tau^2}{\mu} dV$$
$$U^{Torsion} = \frac{1}{2}\iiint \frac{M_t^2 r^2}{\mu \times I_p{}^2} ds dx = \frac{1}{2}\int \frac{M_t^2}{\mu \times I_p{}^2} dx \iint r^2 ds = \frac{1}{2}\int \frac{M_t^2 I_p}{\mu \times I_p{}^2} dx \tag{III-59}$$
$$U^{Torsion} = \frac{1}{2}\int \frac{M_t^2}{\mu \times I_p} dx$$

The deformation energy stored in a shaft with a length L can also be equal to:

$$U^{Torsion} = \frac{1}{2} k_t . \theta^2 \tag{III-60}$$

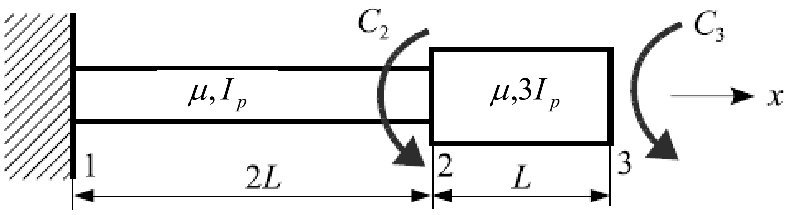
5.3 Castigliano theorem to calculate rotation

The Castigliano theorem allows calculating the rotations θ_{xi}, θ_{yi} or θ_{zi} in a point i respectively around the three axes X, Y or Z of the structure. It is defined as the derivative of the total elastic strain energy with respect to the moment M_i applied in this point i:

$$\theta_{xi} = \frac{\partial U^{Total}}{\partial M_i} \tag{III-61}$$

Example 1:

We consider the x-axis shaft shown in the below figure. Let μ be the transverse modulus of elasticity of the shaft material. The torsion constant is I_p between the points 1 and 2 and $3I_p$ between the points 2 and 3. The point 1 is embedded; the points 2 and 3 carry a respective torque intensities C_2 and C_3 (see the below figure). Let calculate:



1. The expression of the torsion torque $M_t(x)$:

$$\sum M_{/x} = 0 \Rightarrow M_{tx} - C_2 - C_3 = 0 \Rightarrow M_{tx} = C_2 + C_3$$

$$0 < x < 2L \Rightarrow M_t^{12} = M_{tx} = C_2 + C_3$$

$$2L < x < 3L \Rightarrow M_t^{23} = M_{tx} - C_2 = C_2 + C_3 - C_2 = C_3$$

Then:

$M_t^{12} = C_2 + C_3$ and $M_t^{23} = C_3$

2. The total elastic strain energy U^{Total} :

$$U^{Total} = \int_0^{2L} \frac{\left(M_t^{12}\right)^2}{2\mu I_p} dx + \int_{2L}^{3L} \frac{\left(M_t^{23}\right)^2}{2\mu 3 I_p} dx = \frac{L}{\mu I_p}\left[\left(C_2 + C_3\right)^2 + \frac{1}{6}C_3^2\right]$$

3. The rotations θ_{x2} and θ_{x3} respectively in the points 2 and 3:

$$\theta_{x2} = \frac{\partial U^{total}}{\partial C_2} = \frac{L}{\mu I_p}\left(2C_2 + 2C_3\right) \;;\; \theta_{x3} = \frac{\partial U^{total}}{\partial C_3} = \frac{L}{\mu I_p}\left(2C_2 + \frac{7}{3}C_3\right)$$

4. The flexibility and the stiffness matrices of this shaft:

Flexibility matrix $[C]$:

$$\begin{pmatrix}\theta_{x2}\\ \theta_{x3}\end{pmatrix} = \frac{L}{\mu I_p}\begin{bmatrix}2 & 2\\ 2 & 7/3\end{bmatrix}\begin{pmatrix}C_2\\ C_3\end{pmatrix} \Rightarrow [C] = \frac{L}{\mu I_p}\begin{bmatrix}2 & 2\\ 2 & 7/3\end{bmatrix} = \frac{L}{3\mu I_p}\begin{bmatrix}6 & 6\\ 6 & 7\end{bmatrix}$$

Rigidity matrix $[K]$:

$$\begin{pmatrix}C_2\\ C_3\end{pmatrix} = \frac{\mu I_p}{2L}\begin{bmatrix}7 & -6\\ -6 & 6\end{bmatrix}\begin{pmatrix}\theta_{x2}\\ \theta_{x3}\end{pmatrix} \Rightarrow [K] = [C]^{-1} = \frac{\mu I_p}{2L}\begin{bmatrix}7 & -6\\ -6 & 6\end{bmatrix}$$

6. General expression of elastic strain energy

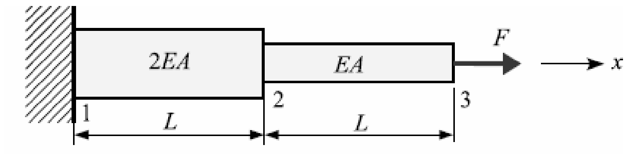
In the case of a combined loadings, the global elastic strain energy for all the loadings (traction, compression, shearing, bending and torsion) is equal to:

$$U^{Global} = \frac{1}{2}\left(\int \frac{F^2}{EA}dx + \int \frac{Mf_z^{\,2}}{EI_z}dx + \int \frac{T_y^{\,2}}{\mu A}dx + \int \frac{Mf_y^{\,2}}{EI_y}dx + \int \frac{T_z^{\,2}}{\mu A}dx + \int \frac{Mt^2}{\mu I_p}dx\right) \quad \text{(III-62)}$$

Directed works No. 3 "Energetic methods for elastic systems"

Exercise N°1

The beam shown in the figure below is embedded at 1. Let E be the Young's modulus of the material. The area of the cross section is 2A between points 1 and 2 and A between points 2 and 3. The beam carries in 3 a force with components (F, 0, 0).



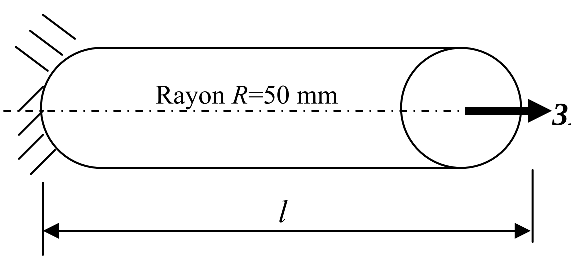
1) Determine the expression for the normal force N(x).

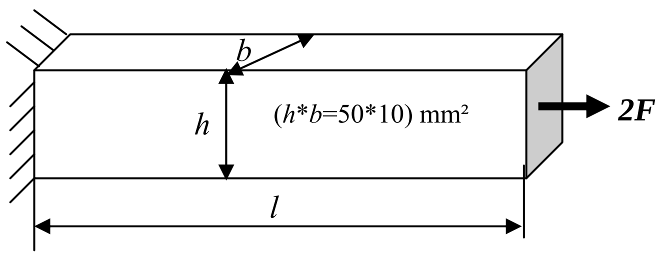
2) If E=200 GPa, A=40mm², L=200 mm and the force F=100 N, then calculate the elastic strain energy and the displacement u_3.

Exercise N°2

Two beams with circular and rectangular cross-sections embedded on their left ends respectively undergo an extension on their right ends by forces with a magnitude of 3F and 2F.

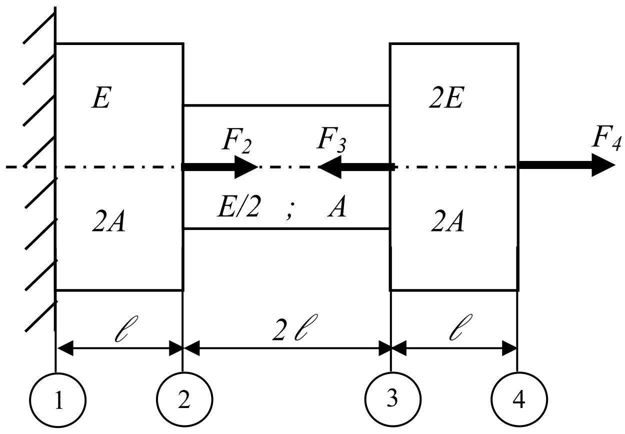
1) If F= 70daN, the length l=2m and the Young's modulus of the material of the two beams E= 210 GPa, calculate the strain energy that the two beams will undergo as well as the displacement of their right ends.





Exercise N°3

For the example of a bar shown in the below figure, establish the expression of the total strain energy $E_{Déf}^{Total}$ and calculate the displacement u_4 if F_2 = 20kN ; F_3 = 20daN ; F_4 = 20N ; ℓ = 2m ; E = 300 GPa ; A = 10 cm².

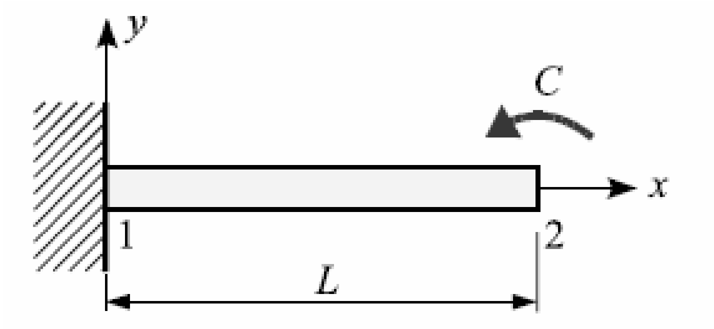


Exercise N°4

The beam below of length L and of constant circular cross section with a diameter d, the beam is embedded at the left point 1. It is made of steel with Young's modulus E. It carries at 2 a torque around the z axis with components (0, 0, C). By neglecting the influence of the shear force (according to the Bernoulli model):

1) Determine the expression for the bending moment *Mf(x)*.

2) Calculate the elastic strain energy knowing that E=200000 MPa, d=30mm, L=0.2 m and the torque C=100 N.m and thus calculate the rotation θ_2.



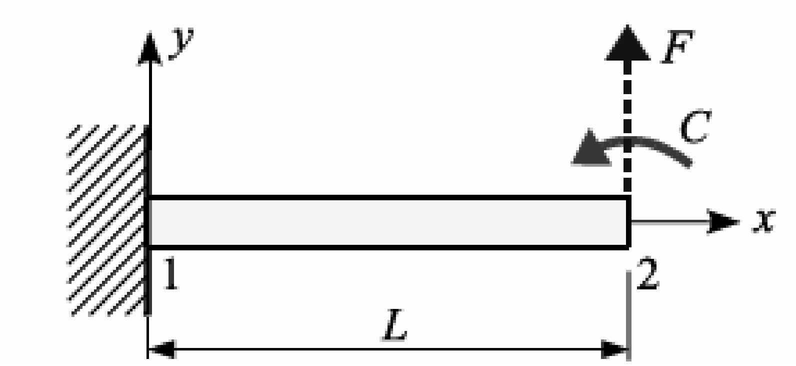
Exercise N°5

The beam presented in the below figure has a length L and a constant circular cross section with a diameter d, the beam is embedded at the left point 1. It is made of steel with Young's modulus E. It carries at 2 a torque with components (0, 0, C) and a force of components (0, F, 0). Neglecting the influence of the shear force, determine:

1) The expression for the bending moment *Mf(x)*.

2) The elastic strain energy knowing that E=200000 MPa, d=30mm, L=0.2 m, F=50 N and the torque C=100 N.m.

3) Using Castigliano's theorem, determine the deflection of the beam at the point 2.



Chapter IV

Combined loadings analysis

Chapter IV: Combined loadings analysis

1. Introduction

In the below figure, we present a typical combined shaft loadings (traction, torsion, symmetrical and unsymmetrical bending) applied at the same time on a shaft.

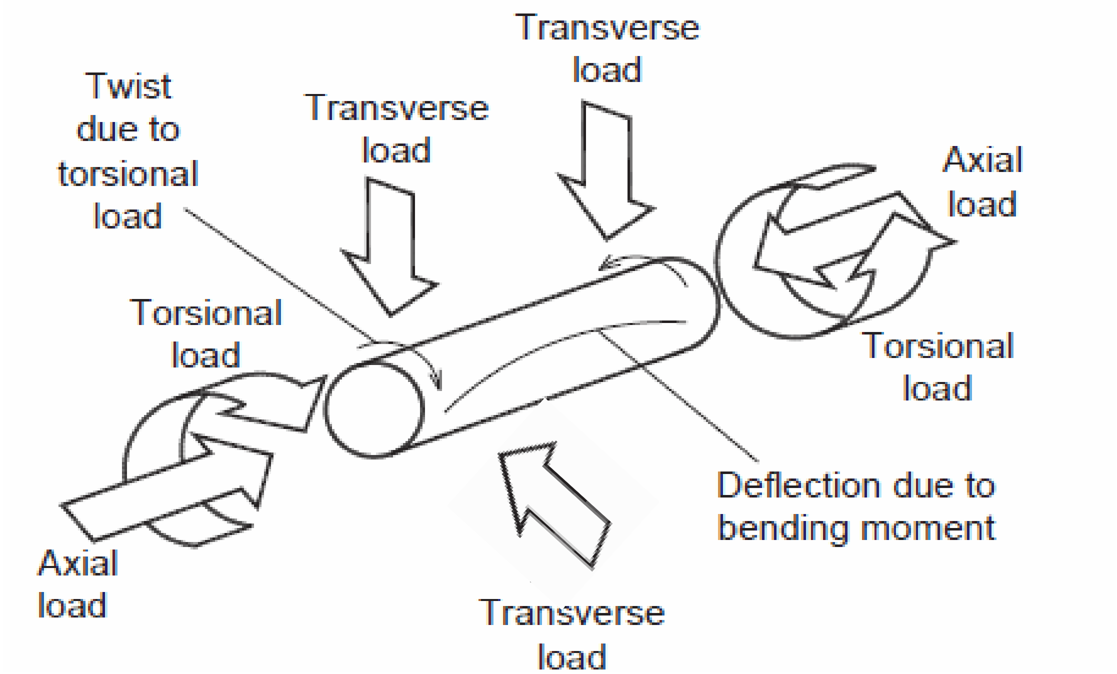


Figure IV- 1: Typical combined shaft loadings [16]

Example:

A shaft of a turbine rotates with a certain rotation speed and with a certain torque which can generate in the event of an anomaly the twisting of the rotating elements, this shaft carries a compressor upstream and a turbine downstream, their weights generate the bending of the shaft, the air sucked in and compressed by the compressor causes the traction of the shaft. Therefore, the shaft simultaneously undergoes a combined solicitation of traction, bending and torsion.

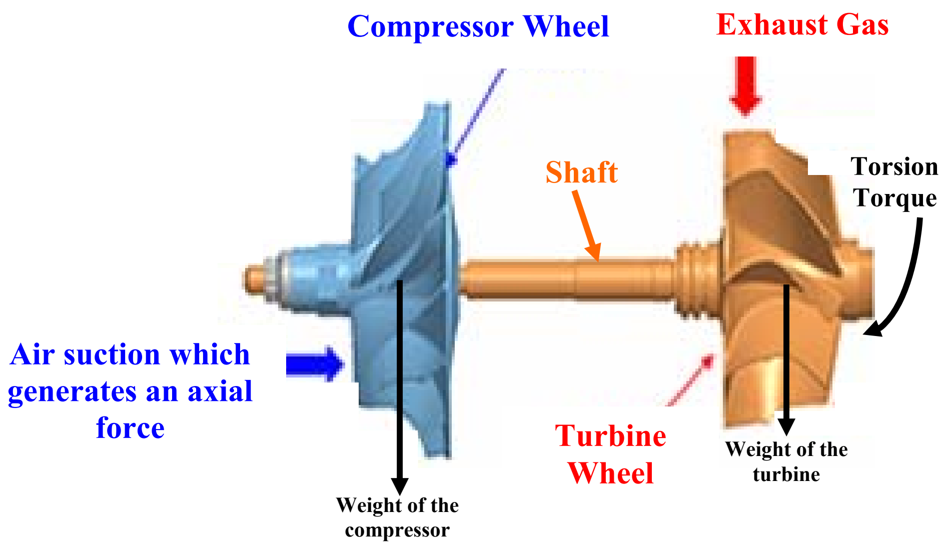


Figure IV- 2: Compressor-turbine assembly [18]

2. Unsymmetrical bending

In plane or symmetrical bending, the loads are applied in the planes of symmetry (xy or xz plane) of the beam. This results that the beam being deformed in a single y or z direction of the symmetrical plane, also called the bending plane or the deflection plane. In this part we will study the non-symmetrical beams and the case of symmetrical beams not loaded in their symmetrical plane; the resulting bending is called unsymmetrical bending.

The below figure shows different symmetrical planes in the cross-sections of different beams:

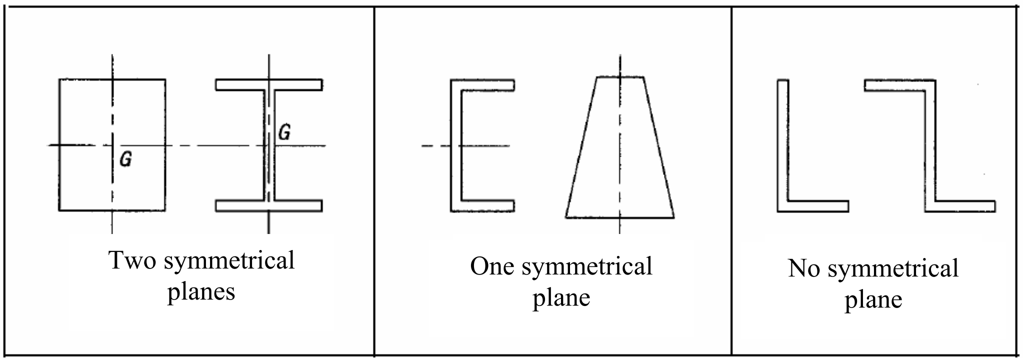
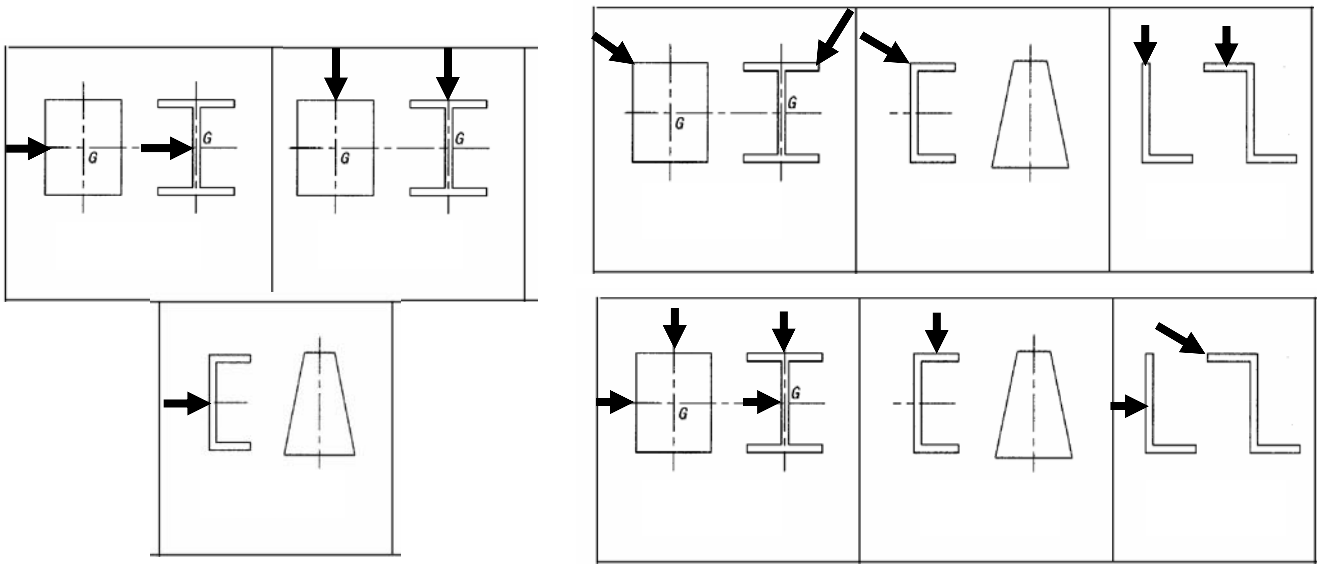


Figure IV- 3: Different planes of symmetry in beams

Note that if we apply a force in one of the symmetrical plane presented in the below figure, we will have a symmetrical bending. Otherwise, if we apply a force in a plane different to the symmetrical plane or we apply simultaneously two forces in two symmetrical planes, we will have an unsymmetrical bending.



(a) Symmetrical bending **(b) Unsymmetrical bending**

Figure IV- 4: Symmetrical and unsymmetrical planes and loadings

The following figure presents the bending moments generated in a beam which has a flexure in the two planes.

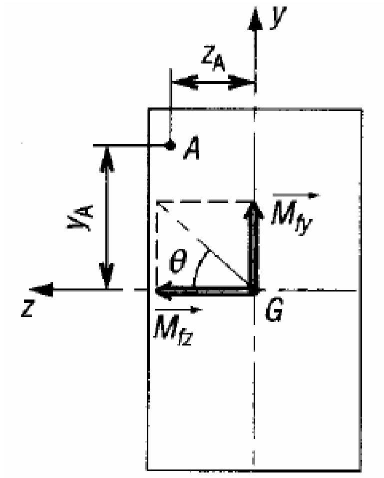


Figure IV- 5: Flexure in two planes

In the above figure, the normal stress due to the unsymmetrical bending is calculated for the A point by the following relationship:

$$\sigma_{xx\,A} = \frac{Mf_y}{I_y} z_A + \frac{Mf_z}{I_z} y_A \qquad \text{(IV-1)}$$

Mf_z is the bending moment around the z axis.
Mf_y is the bending moment around the y axis.
I_z and I_y are the inertia moments of the beam cross-section respectively with reference to z and y axes.

The maximum normal stress caused by an unsymmetrical bending or that is located in unsymmetrical beam is given by the following formula:

$$\sigma_{xx\max} = \frac{Mf_{y\max}}{I_y} z_{\max} + \frac{Mf_{z\max}}{I_z} y_{\max} \qquad \text{(IV-2)}$$

$y_{\max}$ or $z_{\max}$ are the farthest distances from the new neutral plane generated by the unsymmetrical bending.

2.1 Neutral plan and deflection in unsymmetrical bending

The neutral plan corresponds to the plan in which the stresses are equal to zero. At any point (y,z) located in the neutral plan (Figure IV- 6), we can write:

$$\sigma = \frac{Mfy}{Iy} z + \frac{Mfz}{Iz} y = 0 \Rightarrow \frac{Mfy}{Iy} z = -\frac{Mfz}{Iz} y \Rightarrow \frac{y}{z} = \tan\alpha = -\frac{Iz}{Iy}\frac{Mfy}{Mfz} \Rightarrow \alpha = -\arctan\left(\frac{Iz}{Iy}\frac{Mfy}{Mfz}\right)$$

$$\alpha = -\arctan\left(\frac{Iz}{Iy}\frac{Mf\cos\theta}{Mf\sin\theta}\right) = -\arctan\left(\frac{Iz}{Iy}\frac{1}{\tan\theta}\right) = -\arctan\left(\frac{Iz}{Iy} ctg\theta\right) \qquad \text{(IV-3)}$$

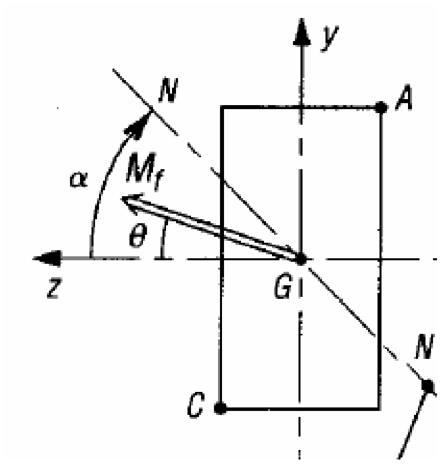


Figure IV- 7: Neutral plan

The θ angle is found between the z axis and the plan of the applying force.

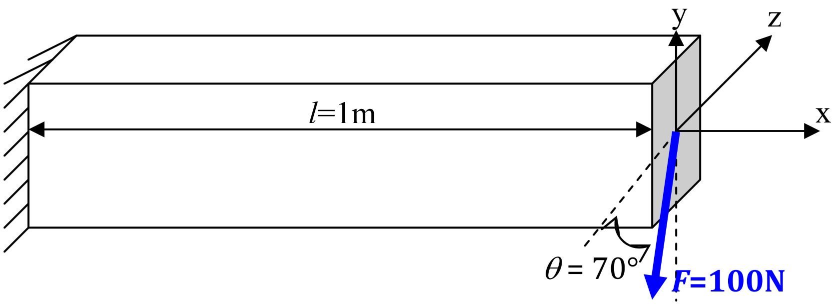
The α angle is found between the z axis and the neutral plan.

The A and C points are the farthest points from the neutral plan, in these points we found the maximum stress of traction or compression.

Example:

The below beam is subjected by a force applied on a plane inclined with respect to the plane of symmetry xz by an angle of 70°. The width of the beam is b=10 mm and its thickness h is equal to 30 mm. Determine:

1) The inertia moments of the beam cross-section I_z and I_y respectively with reference to z and y axes.
2) The maximum bending moments $Mf_{y\,max}$ and $Mf_{z\,max}$.
3) The maximum tensile stress.
4) The angle α of the neutral plane with respect to the z axis and the coordinates of the farthest point from this plane.



1) I_z and I_y

$$Iz = \frac{bh^3}{12} = \frac{10.30^3}{12} = 22500mm^4 = 2{,}25.10^{-8}m^4$$

$$Iy = \frac{hb^3}{12} = \frac{30.10^3}{12} = 2500mm^4 = 2{,}5.10^{-9}m^4$$

2) $Mf_{y\,max}$ and $Mf_{z\,max}$

$$Fy = F\sin\theta = 93{,}97N \;\; ; \;\; Fz = F\cos\theta = 34.2N$$

$$Mf_{zMax} = -Fy * l = -93{,}97N.m$$

$$Mf_{yMax} = -Fz * l = -34{,}2N.m$$

$$Mf_{Max} = \sqrt{\left(Mf_{zMax}^{\;2} + Mf_{yMax}^{\;2}\right)} = 100N.m$$

We can calculate Mf_{max} directly also by:

$$Mf_{Max} = -F * l = -100N.m$$

$$Mf_{zMax} = Mf_{Max}\sin\theta = -100 * \sin 70 = -93{,}97N.m$$

$$Mf_{yMax} = Mf_{Max}\cos\theta = -100 * \cos 70 = -34{,}2N.m$$

3) Maximum tensile stress:

In the calculation, we take the absolute value of the maximum bending moments

$$\sigma_{Max} = \frac{Mf_{yMax}}{Iy} z_{Max} + \frac{Mf_{zMax}}{Iz} y_{Max} = \frac{Mf_{yMax}}{Iy}\frac{b}{2} + \frac{Mf_{zMax}}{Iz}\frac{h}{2} = \frac{34{,}2}{2{,}5.10^{-9}}\frac{0{,}01}{2} + \frac{93{,}97}{2{,}25.10^{-8}}\frac{0{,}03}{2}$$

$$\sigma_{Max} = 1{,}31.10^{8}\,Pa = 131\,MPa$$

The maximum compressive stress is equal to -131 MPa.

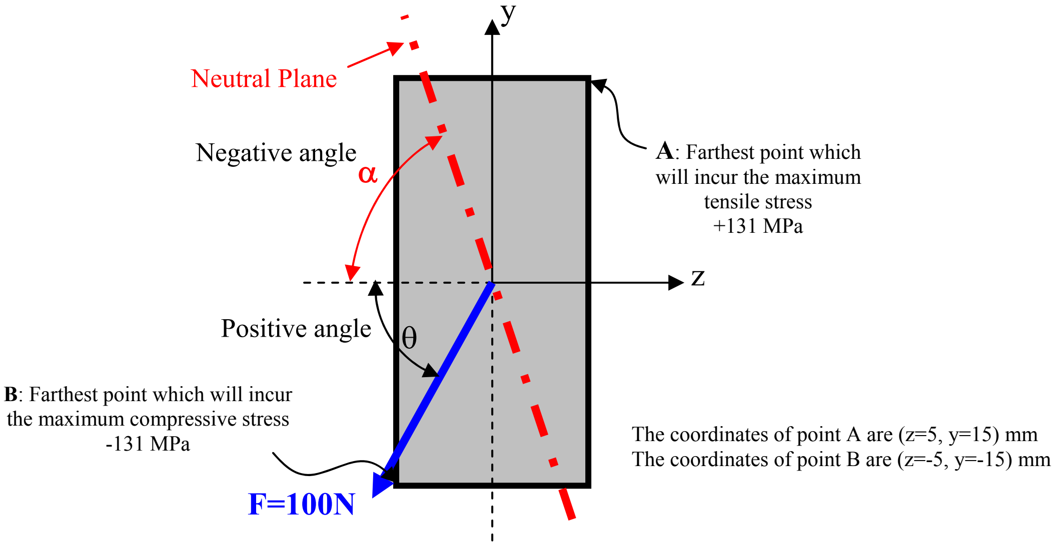
4) The angle α of the neutral plane in relation to the z axis and the coordinates of the point furthest from this plane:

$$\tan\alpha = -\frac{Iz}{Iy}\frac{1}{\tan\theta} \Rightarrow \alpha = -\arctan\left(\frac{Iz}{Iy}\frac{1}{\tan\theta}\right) = -\arctan\left(\frac{2{,}25.10^{-8}}{2{,}5.10^{-9}}\frac{1}{\tan 70}\right) = -73{,}02°$$

Or otherwise:

$$\sigma = \frac{Mfy}{Iy}z + \frac{Mfz}{Iz}y = 0 \Rightarrow \frac{Mfy}{Iy}z = -\frac{Mfz}{Iz}y \Rightarrow \frac{y}{z} = \tan\alpha = -\frac{Iz}{Iy}\frac{Mfy}{Mfz} \Rightarrow \alpha = -\arctan\left(\frac{Iz}{Iy}\frac{Mfy}{Mfz}\right)$$

$$\alpha = -\arctan\left(\frac{Iz}{Iy}\frac{Fz * x}{Fy * x}\right) = -\arctan\left(\frac{Iz}{Iy}\frac{F\cos\theta}{F\sin\theta}\right) = -\arctan\left(\frac{2{,}2510^{-8}}{2{,}510^{-9}} * \frac{34{,}2}{93{,}97}\right) = -73{,}02°$$



3. Tensile with bending loading

We can simply apply an inclined force at the end of the beam and in the plane (xy). This force will be broken down into two forces F_x which generates traction and F_y which generates bending. The below beam will then be subjected to the tensile and bending loadings, the maximum stress generated will be equal to the sum of the maximum stress due to the bending and the maximum stress due to the traction.

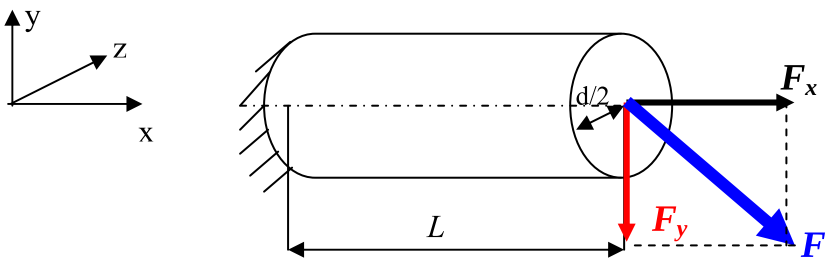


Figure IV- 8: Beam subjected to traction and bending loading

The maximum stress will be equal to:

$$\sigma_{xx\max} = \frac{4F_{traction}}{\pi d^2} + \frac{Mf_{\max}(d/2)}{I_z} \quad \text{with} \quad I_z = \int_S y^2 ds = \frac{\pi d^4}{64} \qquad \text{(IV-4)}$$

The neutral plan in this case is at a distance equal to v' from the axis of the beam, the stress σ is zero in this neutral plane (see the below figure).

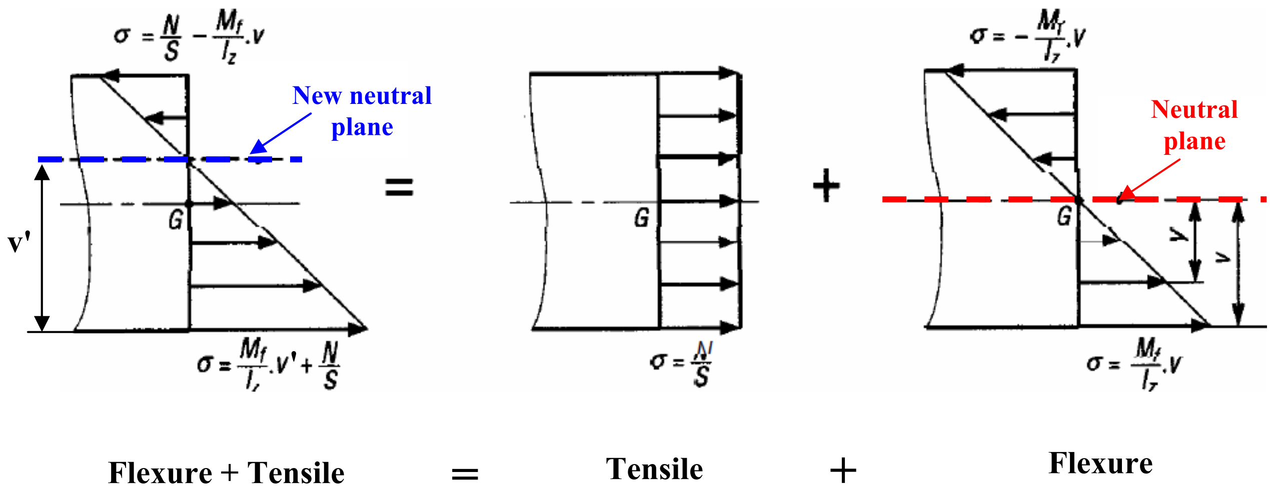


Figure IV- 9: Neutral plane in the tensile-flexure type loading [2]

4. Torsion with bending loading

The shaft presented in the below figure is supported by two supports A and B, it is solicited by flexure and torsion, the motor generates a motor torque *C* which causes a torque in the shaft, the weight *P* of the beam generates its bending.

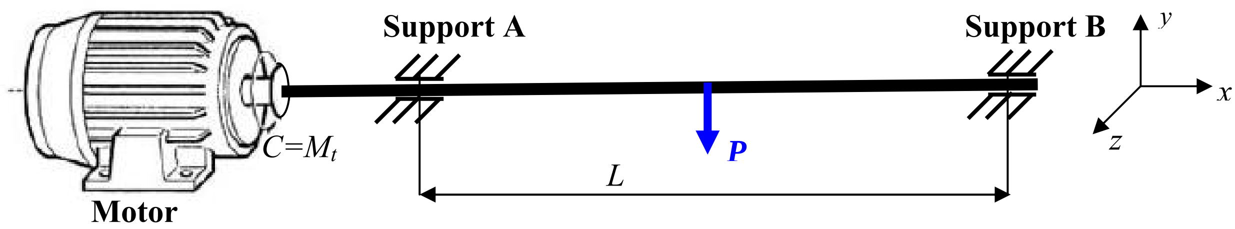


Figure IV- 10: Shaft subjected to torsion and bending loading

The motor power is equal:

$$P_u = C.\omega \quad \text{with } \omega = 2\pi\, N/60 \qquad \text{(IV-5)}$$

P_u is the power motor in Watt.

C is the motor torque in N.m.

ω is the rotation angular speed of the motor in rad/s.

N is the rotation speed of the motor in rpm.

When we design a shaft we must respect several criteria: mechanical resistance criterion, rigidity or deformation criterion, critical speed criterion, fatigue criterion, etc. Thus, it is necessary to avoid unbalancing the shaft and misalignment of the axes. Sometimes, we are forced to optimize the mass of the shaft to reduce the cost of its manufacturing.

The resistance calculation makes it possible to size the shaft by determining the appropriate diameter of the shaft to avoid possible breakage. If the diameter of the shaft is imposed, then we only check the mechanical strength of the shaft.

Whatever the method of calculating the mechanical resistance of the shaft, we must follow the following steps:

1. Calculation of the support reactions.
2. Determine in the vertical and horizontal planes the distribution of the bending moments Mf_z and Mf_y and the torsion moment M_t along the shaft.
3. Add the vertical and horizontal bending moment diagrams to draw the resulting bending moment *Mf* diagram. We thus determine the maximum value of the resulting bending moment and the dangerous section of the shaft.

4. Calculate the maximum equivalent moment Mi such that:

$$Mi = \sqrt{Mf_{\max}^2 + M_{t\max}^2} \quad \text{(IV-6)}$$

5. Determine the distribution of axial loads and their maximum value.

6. Calculate then the maximum normal stress $\sigma_{xx\max}$ due to the bending by the following relationship:

$$\sigma_{xx\max} = \frac{Mf_{\max}(d/2)}{I_z} \quad ; \quad I_z = \frac{\pi d^4}{64} \quad \text{(IV-7)}$$

7. Calculate then the maximum tangential stress $\tau_{xz\ \max}$ due to the torsion using the following relationship:

$$\tau_{xz\ \max} = \frac{M_{t\max}(d/2)}{I_p} \quad ; \quad I_p = 2I_z = \frac{\pi d^4}{32} \quad \text{(IV-8)}$$

8. Finally, we use either the **Von Mises** or **Tresca** criteria to calculate the minimum diameter of the shaft or to check the mechanical strength of the shaft.

The **Von Mises** criterion is generally written like this:

$$(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2) = 2\sigma_{eq}^2 \quad \text{(IV-9)}$$

$$\sigma_{eq} < \sigma_{adm} \quad \text{(IV-10)}$$

The simplified **Von Mises** criterion according to our case is:

$$\sqrt{(\sigma_{xx\max}^2 + 3\tau_{xz\max}^2)} < \sigma_{adm} \quad \text{(IV-11)}$$

The **Tresca** criterion indicates that:

$$\sqrt{(\sigma_{xx\max}^2 + 4\tau_{xz\max}^2)} < \sigma_{adm} \quad \text{(IV-12)}$$

When the traction force is equal to zero, the minimum diameter d of the shaft can be calculated by the following relationship:

$$d > \sqrt[3]{\frac{32Mi}{\pi\sigma_{adm}}} \quad \text{(IV-13)}$$

Example: Combination of flexure and torsion [13]

A speed reducer is made up of two straight toothed gears and two shafts; the assembly is driven by a motor with a power of 15 kW and rotates at a speed of 1000 rpm.

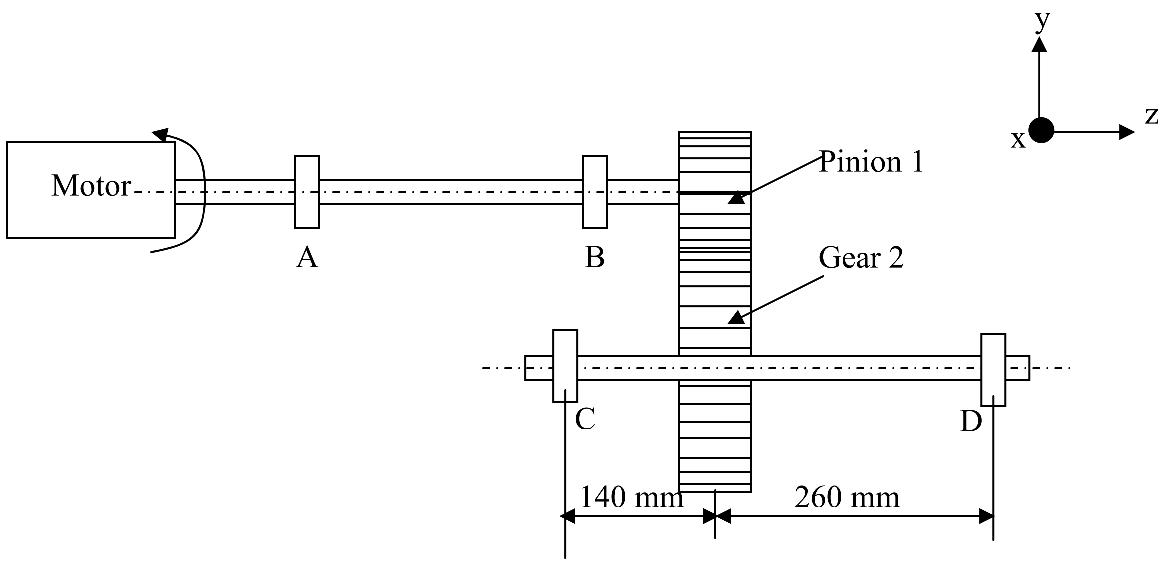
The driving shaft connected to the motor is placed on two bearings A and B and carries the pinion 1 which has a pitch diameter d_1 = 70 mm, while the receiving shaft is supported by two bearings C and D and carries the toothed wheel 2 having a pitch diameter d_2 = 210 mm, the yield stress of the two shafts is σ_Y = 800 MPa, the safety factor *Cs* is equal to 2.

1) Knowing that the pressure angle α = 20°, determine the tangential and radial forces F_t and F_r exerted on the contact teethes of the two gears,

2) Calculate the reactions at the supports C and D.

3) Plot the diagrams of the bending moments M_f and the twisting or torsional moment M_t along the shaft, thus, determine their maximum moments.

4) Determine the minimum diameter of the receiver shaft using the **Tresca** criterion.



Solution:

1) Knowing that the pressure angle α = 20°, determine the tangential and radial forces F_t and F_r exerted on the contact teethes of the two gears,

$P = C\omega$; C is the motor torque (N.m), P =15 kW et ω is the angular speed in rad/s. The rotation speed N is equal to 1000 rpm, and then ω is equal:

$$\omega = \frac{2\pi N}{60} = 104.66 \text{ rad/s}$$

$$C = \frac{P}{\omega} = \frac{15000}{104,66} = 143.31 \text{ N.m}$$

The tangential force F_t applied at the contact of the pinion 1 tooth with the tooth of the gear wheel 2 is calculated by:

$C = F_t * \frac{d_1}{2} \Rightarrow F_t = 2C/d_1$; the pinion 1 have a diameter d_1 = 70 mm

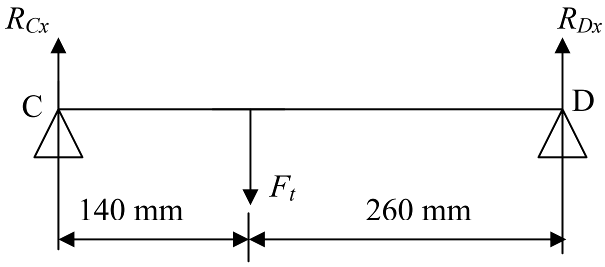
Then, F_t = 4094.57 N

The pressure angle α = 20°, the radial force F_r will be equal:

$$F_r = F_t * tg\alpha = 1490.3 \text{ N}$$

2) Calculus of the C and D support reactions.

R_{Cx} , R_{Dx}

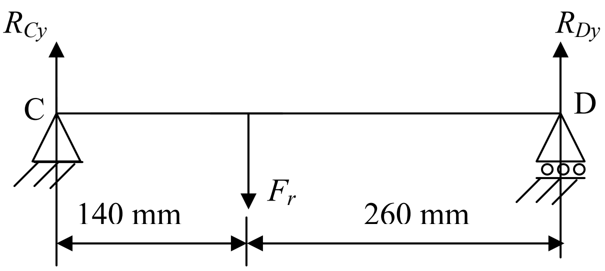


$$\sum F = 0 \Rightarrow R_{Cx} + R_{Dx} - F_t = 0 \quad \ldots\ldots eq(1)$$

$$\sum M_{/C} = 0 \Rightarrow F_t * 0.14 = R_{Dx} * 0,4 \Rightarrow R_{Dx} = \frac{4094,57 * 0,14}{0,4} = 1433.1N$$

$$eq(1) \Rightarrow R_{Cx} = F_t - R_{Dx} = 4094.57 - 1433.1 = 2661.47N$$

R_{Cy} , R_{Dy}



$$\sum F = 0 \Rightarrow R_{Cy} + R_{Dy} - F_r = 0 \quad \ldots\ldots eq(2)$$

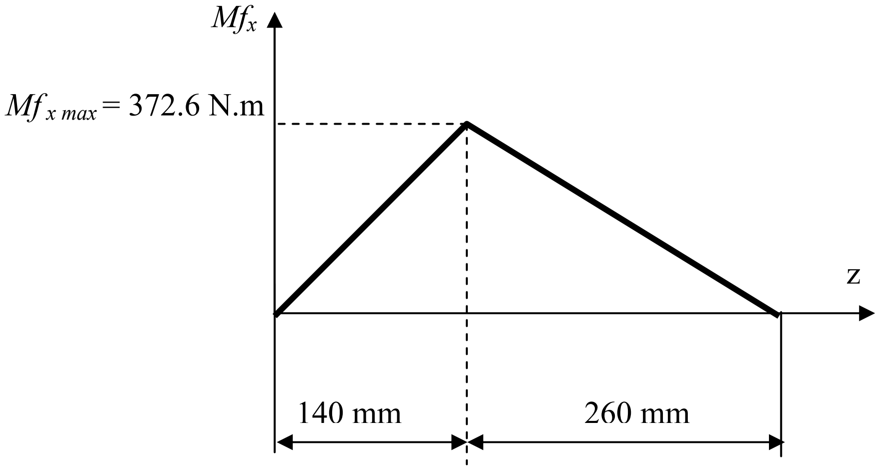
$$\sum M_{/C} = 0 \Rightarrow F_r * 0.14 = R_{Dy} * 0.4 \Rightarrow R_{Dy} = \frac{1490.3 * 0.14}{0.4} = 521.6N$$

$$eq(2) \Rightarrow R_{Cy} = F_r - R_{Dy} = 1490.3 - 1433.1 = 968.7N$$

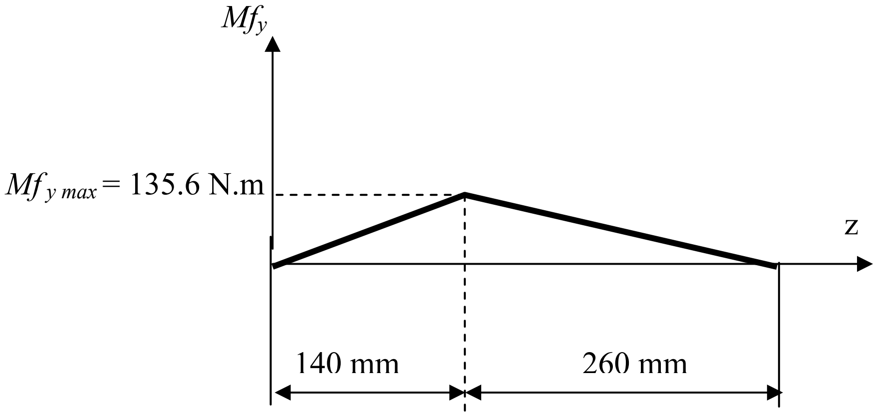
3) Plot the diagrams of the bending moments M_f and the twisting moment M_t along the shaft, thus, determine their maximum moments.

Diagrams of Mf_x et Mf_y:

$Mf_{x\,max} = R_{Cx} \text{ x } 0.14$



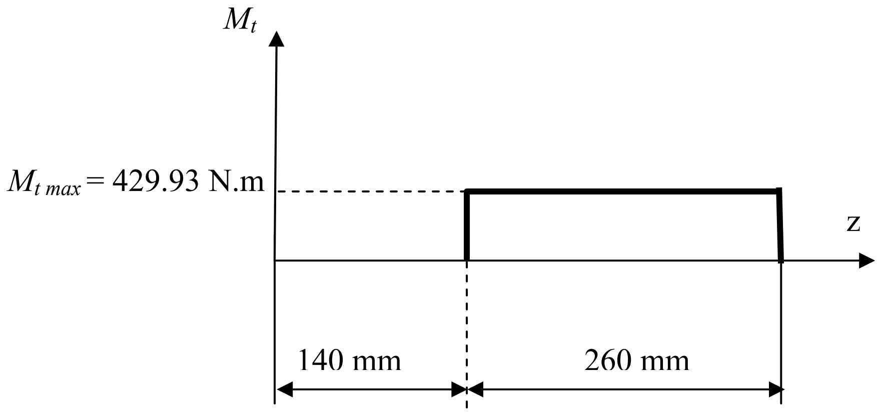
$Mf_{y\,max} = R_{Cy} \text{ x } 0.14$



$$Mf_{\max} = \sqrt{Mf_{x\max}^2 + Mf_{y\max}^2} = 396.5 \text{ N.m}$$

The torsion moment M_t is equal:

$$M_t = F_t \times \frac{d_2}{2} = 4094.57\text{x}0.105 = 429.93 \text{ N.m}$$



4) Déterminer le diamètre minimal de l'arbre récepteur en utilisant le critère de Tresca.

The allowable stress σ_{adm} or the practical resistance R_p of the shaft is equal to σ_Y / Cs ; Cs is the safety coefficient equal to 2 and σ_Y is the shaft material yield stress equal to 800 MPa. Then, the admissible stress or the allowable stress σ_{adm} is equal to 400 MPa.

The **Tresca** criterion is $\sqrt{\sigma_{zz\ \max}^2 + 4\tau_{zx\ \max}^2} < \sigma_{adm}$

$$\sigma_{zz\max} = \frac{Mf_{\max}.d/2}{\pi.d^4/64} \ ; \ \tau_{zx\max} = \frac{M_{t\max}.d/2}{\pi.d^4/32}$$

$$d > \sqrt[3]{\frac{32}{\pi \times \sigma_{adm}}\left(\sqrt{Mf_{\max}^2 + M_{t\max}^2}\right)}$$

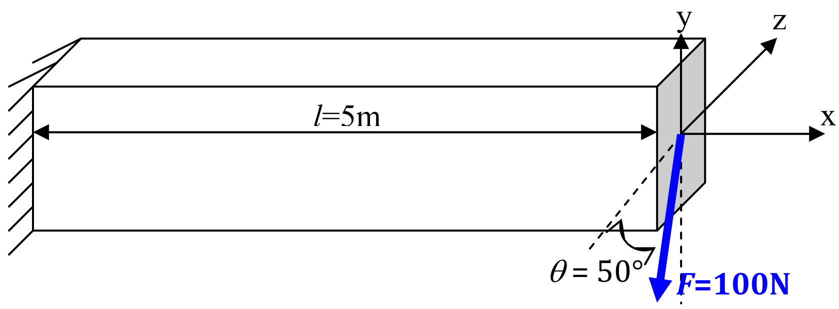
$d > 24.6$ mm

Directed works No. 4 “Combined loadings analysis”

Exercise N°1

The below beam is subjected by a force applied on a plane inclined with respect to the plane of symmetry xz by an angle of 50°. The width of the beam is b=30 mm and its thickness h is equal to 60 mm. Determine:

1) The inertia moments of the beam cross-section I_z and I_y respectively with reference to z and y axes.
2) The maximum bending moments Mf_{ymax} and Mf_{zmax}.
3) The maximum tensile stress.
4) The angle α of the neutral plane with respect to the z axis and the coordinates of the farthest point from this plane.



Exercise N°2

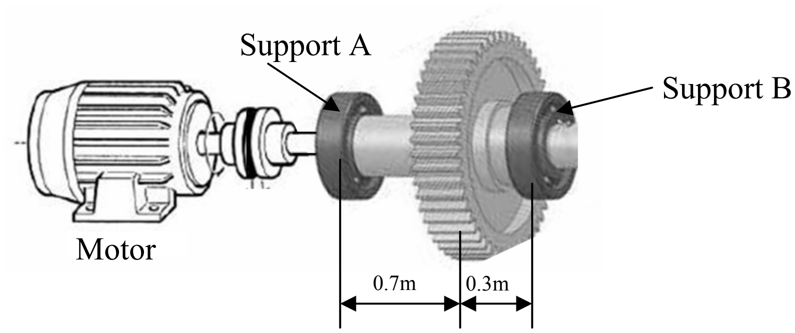
The stranded cable below is used to maintain a lifting system which is not shown in the below figure, the cable weighs 1650 kg, the cable is considered to have a circular section with a diameter d=15 cm, the length of the cable is 12 m, With the help of a tensioner, the cable is stretched by a tensile force equal to 10 kN.

1) Calculate the maximum stress generated in the cable.

Exercise N°3

A motor with a power of 1.5 kW rotates with an angular speed of 3000 rpm, the motor turns a shaft mounted on two simple supports A and B (see the below figure), the shaft carries a toothed wheel which weighs 10 kg and rotates with the same speed as the motor. The material of the shaft is steel, its allowable stress σ_{adm} =400 MPa.

1) Draw the bending and twisting moment diagrams and deduce the diameter of the shaft.



Chapter V

Solution of hyperstatic structures

Chapter V: Solution of hyperstatic structures

1. Introduction

A system, or a beam, is said to be hyperstatic whenever the reactions exerted by the connections cannot be calculated from the fundamental equations of static $\Sigma F=0$ & $\Sigma M=0$. The reactions can only be determined after writing other equations obtained from the deformations of the system. Several methods are then used to solve hyperstatic systems.

2. Example of hyperstatic systems:

Example 1:

The following figure concerns a bridge used for loading merchandises. The maximum allowable load is F_{max}. A circular cross-section profile was used for the three bars and a rectangular profile was used for the beam that carries the load. The system has 3 supports, so it is considered as hyperstatic system.

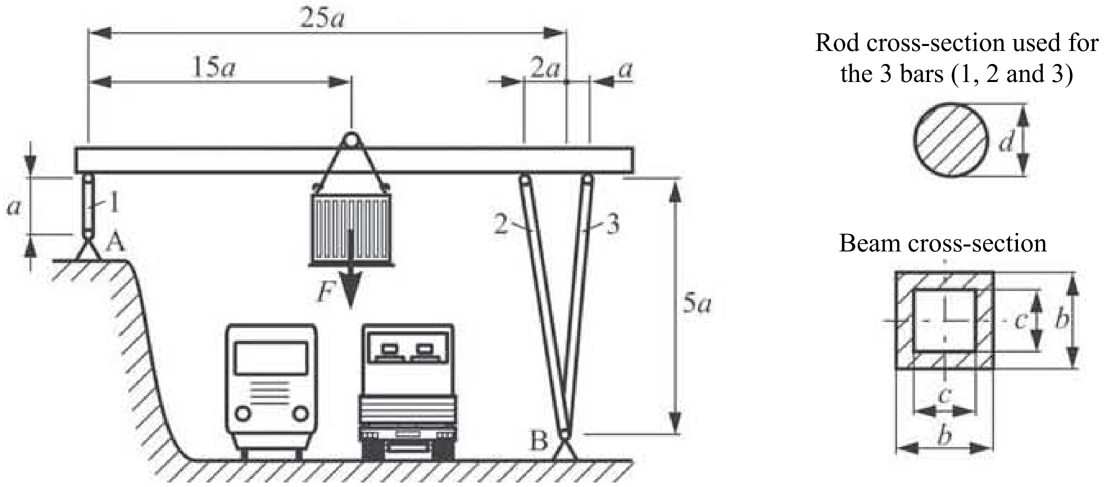


Figure V- 1: Beam system for carrying loads [5]

In the following example, the action of the air on the wing is schematized by a distributed load q. The airplane wing AC is embedded in the airplane cabin and supported by an undeformable bar BD. The embedding moment at the point C and the actions exerted by the two supports B and C make this wing hyperstatic.

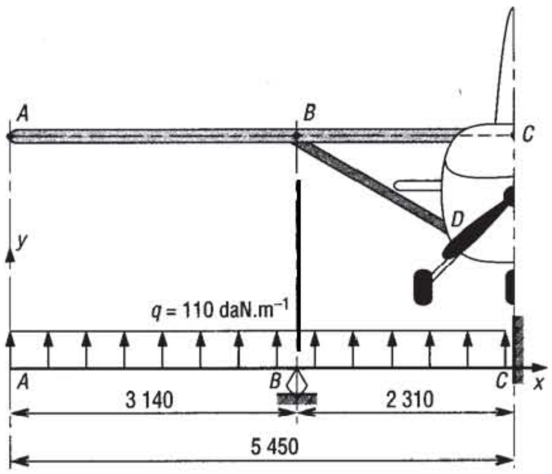


Figure V- 2: Hyperstatic wing [2]

Example 2:

We like to calculate for the following Example 2 the embedding moment M_A at the A point and the reactions R_A and R_B respectively at the embedding point A and in the support point B.

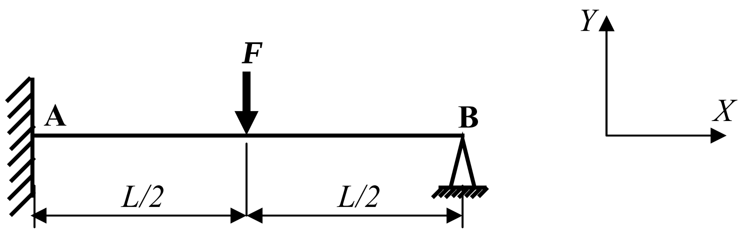


Figure V- 3: Cantilever beam supported by an additional B support at the right extremity

The unknown moment and reactions are presented in the following figure:

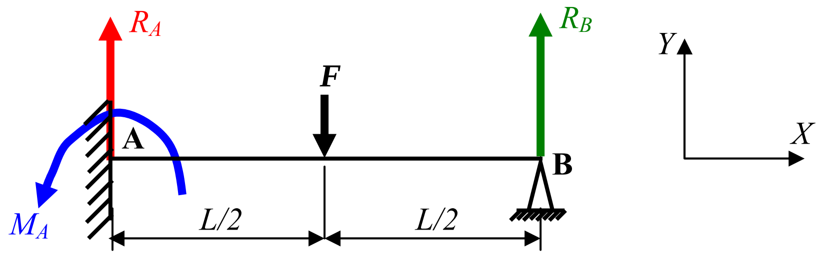


Figure V- 4: Unknown moment and reactions

Using the static equilibrium equations, we have:

$$\sum F_{/Y} = 0 \Rightarrow R_A + R_B - F = 0 \Rightarrow R_A + R_B = F$$

$$\sum M_{/A} = 0 \Rightarrow M_A + R_B L - F\frac{L}{2} = 0$$

So, we have 2 equations and 3 unknowns (M_A, R_A and R_B), the previous system of equations cannot be solved, it is called hyperstatic system. To solve the previous equation, we should use the calculation methods of the beam curvature seen previously as the integration method, the superposition method, the energetic method; we can use also other method as the force method, etc.

3. Degree of hyperstaticity

The degree of hyperstaticity N is defined as:

$$N = U - S \qquad \text{(V-1)}$$

U is the number of unknowns

S is the number of the static equilibrium equations

To classify a structure if it is isostatic, labily or hyperstatic in the most of cases is enough count and evaluate the following difference:

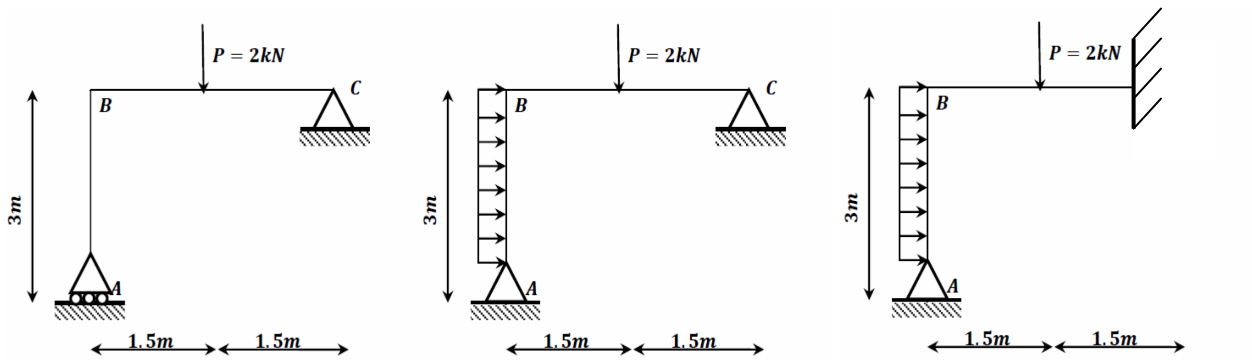
$N - S < 0 \quad \Rightarrow \quad$ Hypestatic system

$N - S = 0 \quad \Rightarrow \quad$ Isostatic system

$N - S > 0 \quad \Rightarrow \quad$ Labile system

The second equation can implies lability and/or isostatic condition, in this case is necessary a deeper study of the structure to understand which category belongs the structure.

<u>Examples:</u>



N=3-3=0
Isostatic system

N=4-3=1
Hyperstatic system

N=5-3=2
Hyperstatic system

Figure V- 5: Degree of hyperstaticity *N* for some systems

4. Solution of hyperstatic structures

4.1. Integration method to obtain the beam curvature equation and to resolve the hyperstatic system

The fundaments of the method is based on the calculation using the integration method the deflection at the support when the unknown reaction was applied and then we make this deflection zero and we will deduce its value.

Let calculate the unknown reactions and moment for the previous hyperstatic system shown in the Figure V- 3. We remove the support B and we calculate firstly, the deflection of the B point as is indicated in the following Figure V- 6:

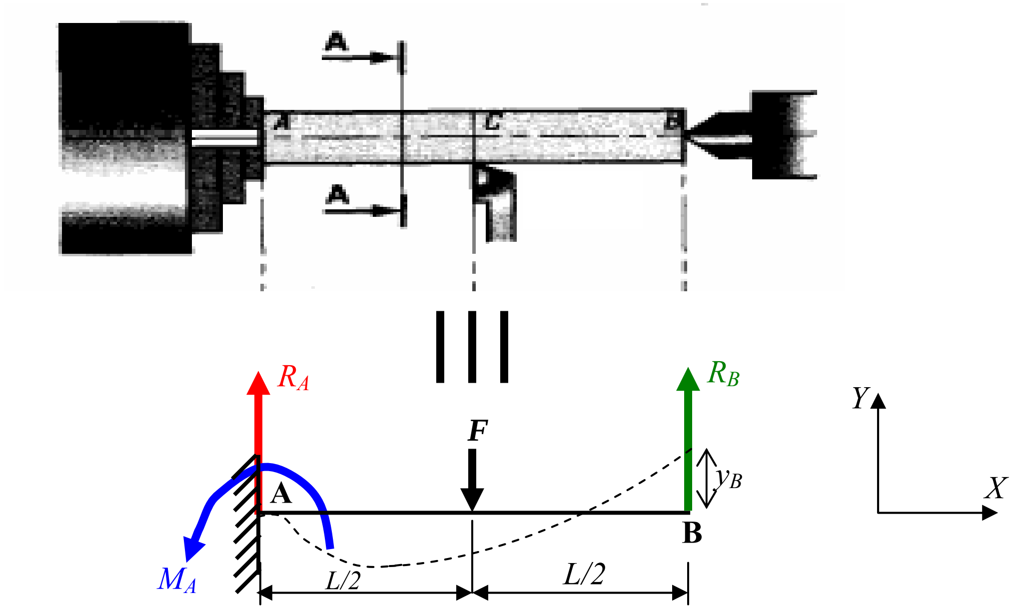


Figure V- 6: The deflection of the hyperstatic cantilever beam

Static equilibrium equations:

Using the static equilibrium equations, we have obtained:

$$\sum F_{/Y} = 0 \Rightarrow R_A + R_B - F = 0 \Rightarrow R_A + R_B = F$$

$$\sum M_{/A} = 0 \Rightarrow M_A + R_B L - F\frac{L}{2} = 0 \Rightarrow M_A + R_B L = F\frac{L}{2}$$

Bending moments:

$$0 < x < L/2 \Rightarrow Mf_1(x) = R_A x - M_A$$

$$L/2 < x < L \Rightarrow Mf_2(x) = R_A x - M_A - F\left(x - \frac{L}{2}\right) = (R_A - F)x - \left(M_A - F\frac{L}{2}\right)$$

Beam deflection by integration method:

$$\frac{d^2 y(x)}{dx^2} = \frac{Mf(x)}{EI_z} \;\ldots\ldots eq(1)$$

Knowing that, $\frac{dy(x)}{dx} = \theta(x)$

$$eq(1) \Rightarrow \frac{d\theta(x)}{dx} = \frac{Mf(x)}{EI_z} \Rightarrow \theta(x) = \int \frac{Mf(x)}{EI_z} dx$$

And then, $y(x) = \int \theta(x)\, dx$

For $0 < x < L/2$, we have:

$$\theta_1(x) = \int \frac{Mf_1(x)}{EI_z} dx = \frac{1}{EI_z} \int (R_A x - M_A) dx$$

$$\Rightarrow \theta_1(x) = \frac{1}{EI_z}\left(\frac{1}{2} R_A x^2 - M_A x + C_1\right)$$

$$\Rightarrow y_1(x) = \int \theta_1(x)\, dx = \frac{1}{EI_z}\left(\frac{1}{6} R_A x^3 - \frac{1}{2} M_A x^2 + C_1 x + C_2\right)$$

C_1 and C_2 are constants to be determined by the boundary conditions.

We know that in the embedding point A, we have:

$y_1(0) = 0 \Rightarrow C_2 = 0$

$\theta_1(0) = 0 \Rightarrow C_1 = 0$

$$\Rightarrow \theta_1(x) = \frac{1}{EI_z}\left(\frac{1}{2} R_A x^2 - M_A x\right)$$

$$\Rightarrow y_1(x) = \frac{1}{EI_z}\left(\frac{1}{6} R_A x^3 - \frac{1}{2} M_A x^2\right)$$

For $L/2 < x < L$, we have:

$$\theta_2(x) = \int \frac{Mf_2(x)}{EI_z} dx = \frac{1}{EI_z} \int \left[(R_A - F)x - \left(M_A - F\frac{L}{2}\right)\right] dx$$

$$\Rightarrow \theta_2(x) = \frac{1}{EI_z}\left(\frac{1}{2}(R_A - F)x^2 - \left(M_A - F\frac{L}{2}\right)x + C_3\right)$$

$$\Rightarrow y_2(x) = \int \theta_2(x)\, dx = \frac{1}{EI_z}\left(\frac{1}{6}(R_A - F)x^3 - \frac{1}{2}\left(M_A - F\frac{L}{2}\right)x^2 + C_3 x + C_4\right)$$

C_3 and C_4 are constants to be determined by the boundary conditions.

In *x=L/2*, we have:

$\theta_1(L/2) = \theta_2(L/2)$ and $y_1(L/2) = y_2(L/2)$

If $\theta_1(L/2) = \theta_2(L/2)$, then we have:

$$\Rightarrow \theta_1(L/2) = \frac{L}{2EI_z}\left(\frac{1}{4} R_A L - M_A\right) = \theta_2(L/2) = \frac{L}{2EI_z}\left(\frac{1}{4}(R_A - F)L - \left(M_A - F\frac{L}{2}\right) + \frac{2}{L} C_3\right)$$

$$\Rightarrow C_3 = -F\frac{L^2}{8}$$

$$\Rightarrow \theta_2(x) = \frac{1}{EI_z}\left(\frac{1}{2}(R_A - F)x^2 - \left(M_A - F\frac{L}{2}\right)x - F\frac{L^2}{8}\right)$$

If $y_1(L/2)=y_2(L/2)$, then we have:

$$\Rightarrow y_1(L/2)=\frac{1}{EI_z}\left(\frac{1}{48}R_A L^3-\frac{1}{8}M_A L^2\right)=y_2(L/2)=$$

$$=\frac{1}{EI_z}\left(\frac{1}{48}(R_A-F)L^3-\frac{1}{8}\left(M_A-F\frac{L}{2}\right)L^2-F\frac{L^3}{16}+C_4\right)$$

$$\Rightarrow C_4=+F\frac{L^3}{48}$$

$$\Rightarrow y_2(x)=\frac{1}{EI_z}\left(\frac{1}{6}(R_A-F)x^3-\frac{1}{2}\left(M_A-F\frac{L}{2}\right)x^2-F\frac{L^2}{8}x+F\frac{L^3}{48}\right)$$

In fact, in reality $y_2(L)=y_B=0$ because we have a support B at $x=L$, then:

$$\Rightarrow y_2(L)=\frac{1}{EI_z}\left(\frac{1}{6}(R_A-F)L^3-\frac{1}{2}\left(M_A-F\frac{L}{2}\right)L^2-F\frac{L^2}{8}L+F\frac{L^3}{48}\right)=0$$

$$\Rightarrow \frac{1}{3}R_A L^3-M_A L^2=\frac{1}{24}FL^3$$

Now, we have three equations with three unknowns R_A, R_B and M_A.

$$\Rightarrow \begin{cases} R_A+R_B=F \\ M_A+R_B L=F\dfrac{L}{2} \\ \dfrac{1}{3}R_A L^3-M_A L^2=\dfrac{1}{24}FL^3 \end{cases}$$

The solution of the above system of equations gives:

$$M_A=+\frac{3}{16}FL$$

$$R_A=\frac{1}{8}F+\frac{3}{L}M_A=+\frac{11}{16}F$$

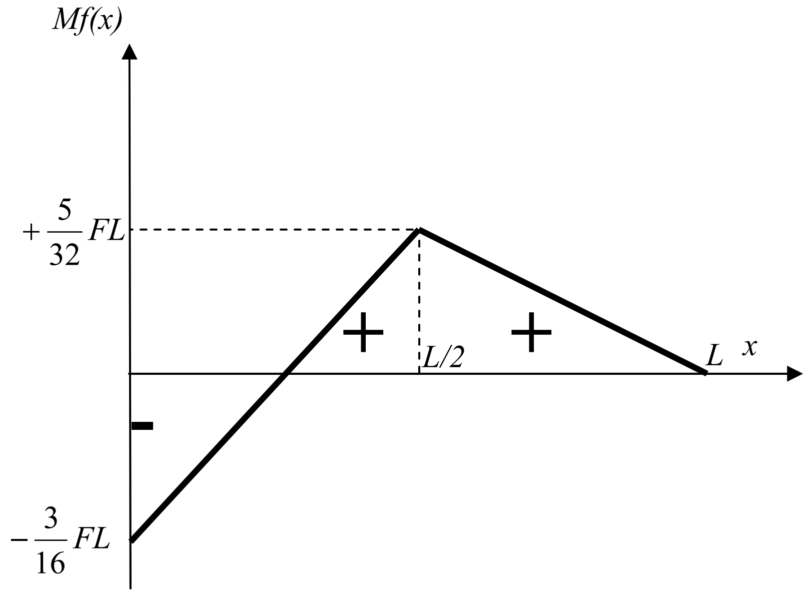
$$R_B=\frac{1}{2}F-\frac{1}{L}M_A=F-R_A=+\frac{5}{16}F$$

We can now replace the obtained values of the reactions R_A and R_B and the moment M_A in the bending moment equations $Mf_1(x)$ and $Mf_2(x)$ to plot the bending moment diagram along the beam, and in the deflection equations $y_1(x)$ and $y_2(x)$ to schematize the beam curvature.

Bending moment diagram:

$$0 < x < L/2 \Rightarrow Mf_1(x) = R_A x - M_A = (F/16)(11x - 3L)$$

$$L/2 < x < L \Rightarrow Mf_2(x) = (R_A - F)x - \left(M_A - F\frac{L}{2}\right) = (F/16)(-5x + 5L)$$



The maximum bending moment Mf_{max} is located at the embedding zone and is equal to $+\frac{3}{16}FL$.

Curvature of the beam deflection:

$$y_1(x) = \frac{1}{EI_z}\left(\frac{1}{6}R_A x^3 - \frac{1}{2}M_A x^2\right) = \frac{F}{96EI_z}\left(11x^3 - 9Lx^2\right)$$

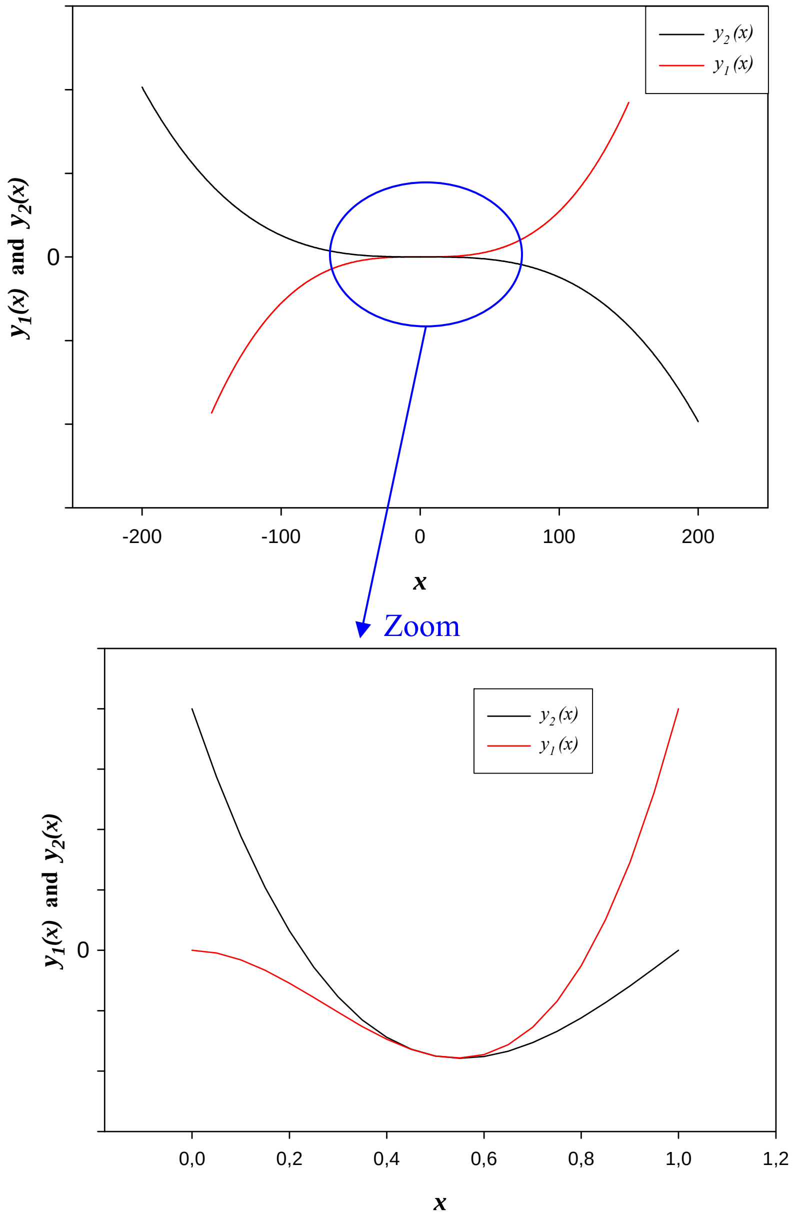
$$y_2(x) = \frac{1}{EI_z}\left(\frac{1}{6}(R_A - F)x^3 - \frac{1}{2}\left(M_A - F\frac{L}{2}\right)x^2 - F\frac{L^2}{8}x + F\frac{L^3}{48}\right)$$
$$= \frac{F}{96EI_z}\left(-5x^3 + 15Lx^2 - 12L^2x + 2L^3\right)$$

We can use Matlab [6] or SigmaPlot 14.0 to plot the previous deflection equations $y_1(x)$ and $y_2(x)$.

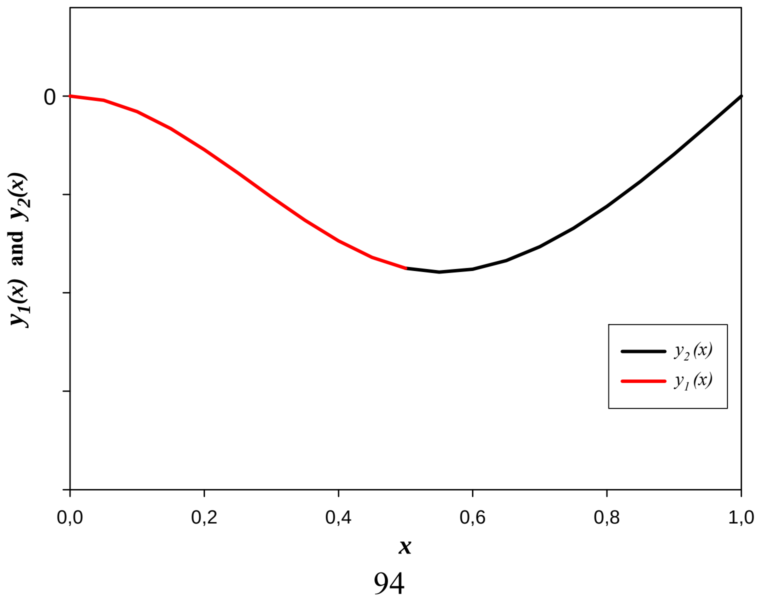
Matlab Program:

```
ezplot('11*x^3-9*x^2')

ezplot('-5*x^3+15*x^2-12*x+2')

Or
x=-200:0.2:+200
y1=11*x.^3-9*x.^2

y2=-5*x.^3+15*x.^2-12*x+2

plotyy(x,y1,x,y2,'plot');
```

The curves of $y_1(x)$ and $y_2(x)$ are presented in the same below graph:

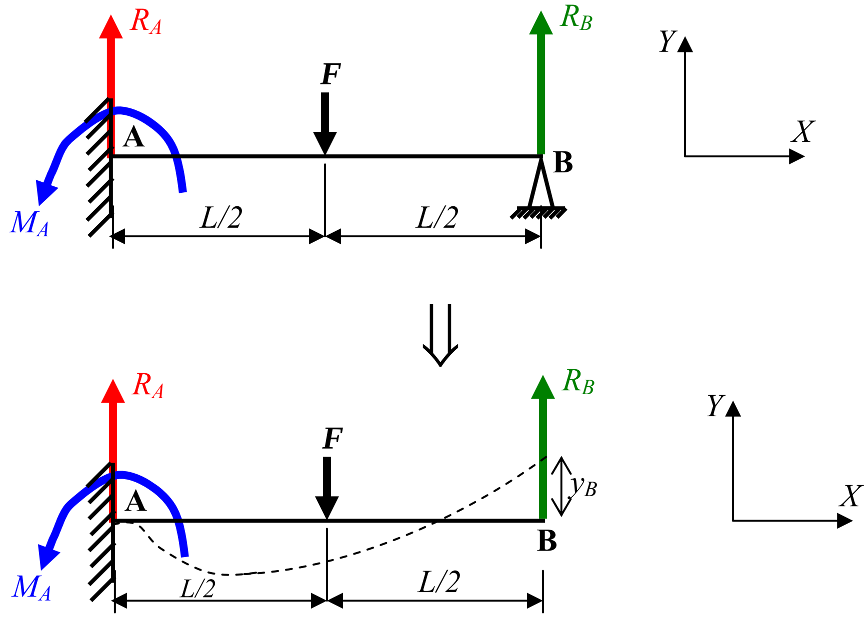


We interest only to x varies from 0 to L. Consequently, the curvature of the beam deflection will have the following shape with a minimal deflection located at x=0.553L.



4.2. Energetic method to calculate the deflection and to resolve the hyperstatic system

Like before, we remove the support B.



Static equilibrium equations:

Using the static equilibrium equations, we have obtained:

$$\sum F_{/Y} = 0 \Rightarrow R_A + R_B - F = 0 \Rightarrow R_A + R_B = F$$

$$\sum M_{/A} = 0 \Rightarrow M_A + R_B L - F\frac{L}{2} = 0 \Rightarrow M_A + R_B L = F\frac{L}{2}$$

Bending moments:

$$0 < x < L/2 \Rightarrow Mf_1(x) = R_A x - M_A = (F - R_B)x + L\left(R_B - \frac{F}{2}\right)$$

$$L/2 < x < L \Rightarrow Mf_2(x) = R_A x - M_A - F\left(x - \frac{L}{2}\right) = (R_A - F)x - \left(M_A - F\frac{L}{2}\right)$$

$$= -R_B x + R_B L = R_B(L - x)$$

The total elastic strain energy of bending U^{Total}:

If we neglect the energy due to the shearing, the bending elastic strain energy is equal in general to:

$U^{Bending} = \frac{1}{2}\int \frac{Mf_z^2}{EI_z}dx$, then:

$$U^{Total} = \frac{1}{2EI_z}\left(\int_0^{L/2} Mf_1^2 dx + \int_{L/2}^{L} Mf_2^2 dx\right)$$

$$U^{Total} = \frac{1}{2EI_z}\left(\int_0^{L/2}\left[(F - R_B)x + L\left(R_B - \frac{F}{2}\right)\right]^2 dx + \int_{L/2}^{L}(-R_B x + R_B L)^2 dx\right)$$

$$U^{Total} = \frac{L^3}{48EI_z}\left(8R_B^2 - 5FR_B + F^2\right)$$

The deflection y_B at the B point calculated by the energetic method:

$$y_B = \frac{\partial U^{total}}{\partial R_B} = \frac{L^3}{48EI_z}\left(16R_B - 5F\right)$$

In fact, in reality $y_B = 0$ because we have a support B at $x=L$, then:

$$y_B = 0 = \frac{L^3}{48EI_z}\left(16R_B - 5F\right)$$
$$\Rightarrow \left(16R_B - 5F\right) = 0$$
$$\Rightarrow R_B = +\frac{5}{16}F$$

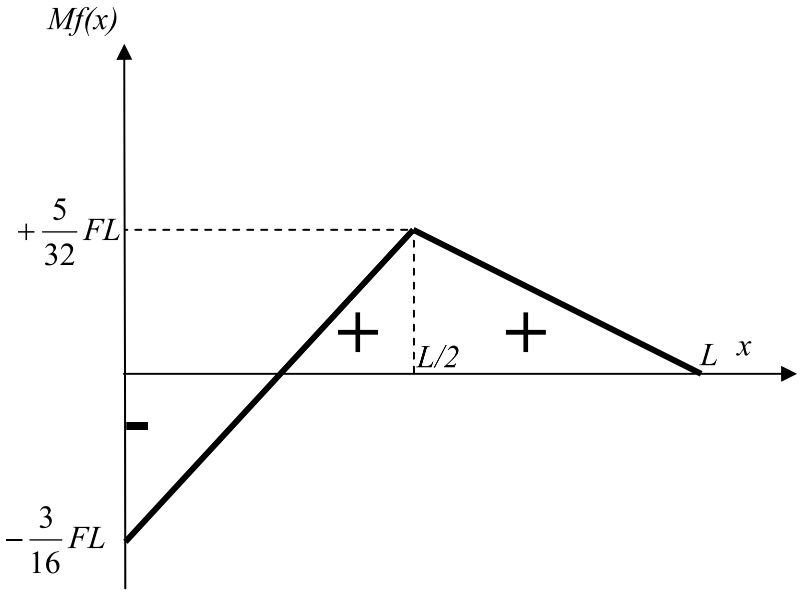
From the previous static equilibrium equations, we have:

$$R_A + R_B = F \Rightarrow R_A = F - R_B = F - \frac{5}{16}F = \frac{11}{16}F$$
$$\Rightarrow R_A = +\frac{11}{16}F$$
$$M_A + R_B L = F\frac{L}{2} \Rightarrow M_A = F\frac{L}{2} - R_B L = \frac{1}{2}FL - \frac{5}{16}FL = \frac{3}{16}FL$$
$$\Rightarrow M_A = +\frac{3}{16}FL$$

With the energetic method, we obtain the same three unknowns R_A, R_B and M_A previously calculated with the **integration method**. We can now replace the obtained values of the reactions R_A and R_B and the moment M_A in the bending moment equations $Mf_1(x)$ and $Mf_2(x)$ to plot the bending moment diagram along the beam.

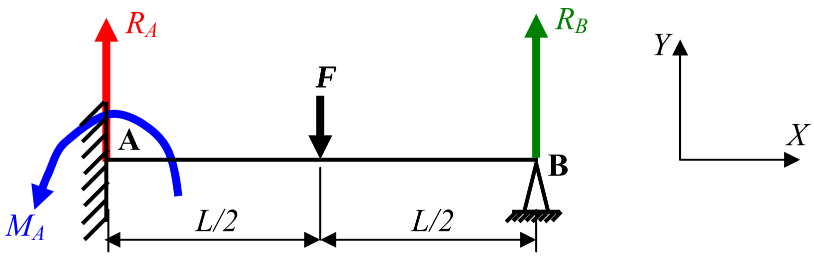
Bending moment diagram:

$$0 < x < L/2 \Rightarrow Mf_1(x) = R_A x - M_A = (F/16)(11x - 3L)$$
$$L/2 < x < L \Rightarrow Mf_2(x) = \left(R_A - F\right)x - \left(M_A - F\frac{L}{2}\right) = (F/16)\left(-5x + 5L\right)$$



4.3. Numerical calculation of the previous hyperstatic example and comparison of the numerical results with the several theoretical results obtained previously

If we take in the following example F=100000 N and L=1m; the beam has a radius R=50 mm and was made of steel with a **Young's** modulus E equal to 210 GPa.



a) Theoretical calculus of the unknowns M_A, R_A, R_B, the maximum bending Mf_{max} and the minimal deflection of the beam y_{min}.

With the help of the two previous theoretical methods, we know that:

$$M_A = +\frac{3}{16}FL = 18750 \text{ N.m.}$$

$$R_A = \frac{1}{8}F + \frac{3}{L}M_A = +\frac{11}{16}F = 68750 \text{ N.}$$

$$R_B = \frac{1}{2}F - \frac{1}{L}M_A = F - R_A = +\frac{5}{16}F = 31250 \text{ N.}$$

$$Mf_{\max} = +\frac{3}{16}FL = 18750 \text{ N.m.}$$

As it was indicated precedently, the minimal deflection y_{min} is located at x=0.553L=0.553m, we can calculate y_{min} using the following equation

$$y_2(x) = \frac{F}{96EI_z}\left(-5x^3 + 15Lx^2 - 12L^2x + 2L^3\right) \text{ with } I_z = \frac{\pi R^4}{4}$$

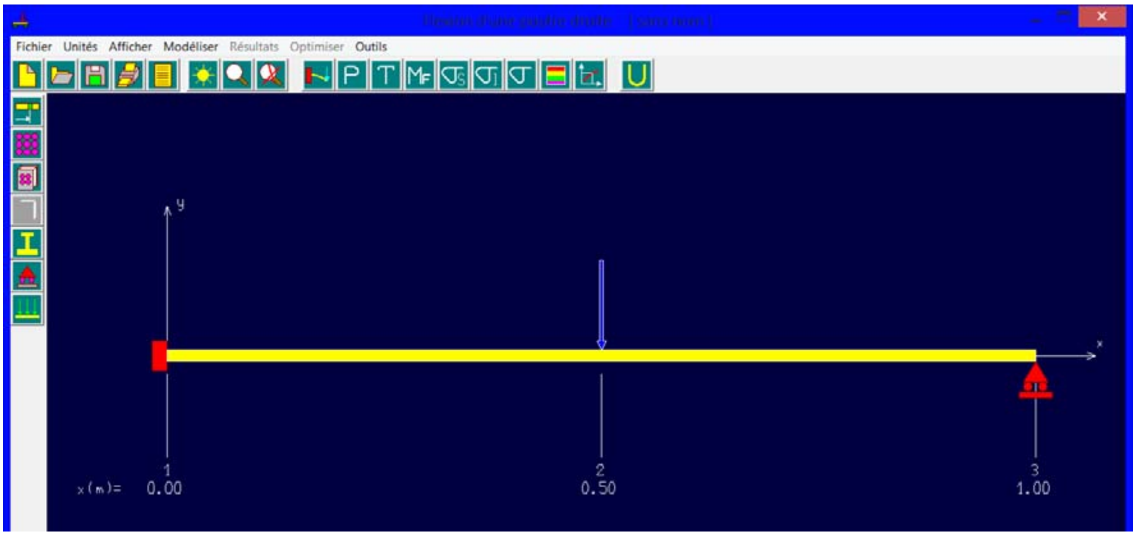
Digital Application (*App*):

$I_z = 4.9\times10^{-6}\,\text{m}^4$ and $y_{\min} = y_2(0.553) = -9.042\times10^{-4}$ m=-0.9042mm.

b) Numerical results (obtained by *RDM6*) of the unknowns M_A, R_A, R_B, the maximum bending Mf_{max} and the minimal deflection of the beam y_{min}.

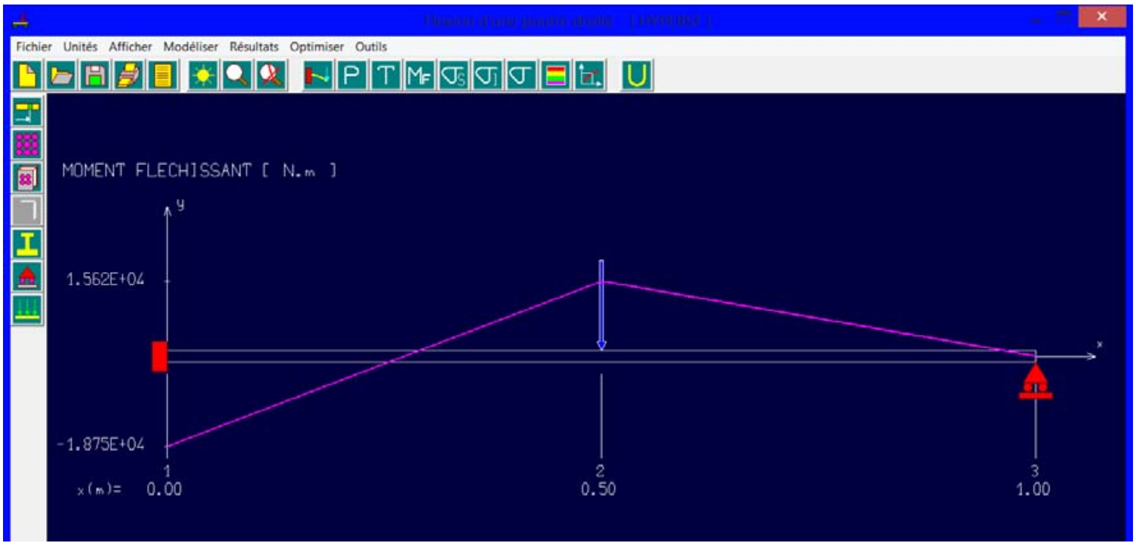
Using the *RDM6* software [7], we could plot the variation of the bending moment and the beam deflection (see the below figures). A very good agreement was found between the theoretical results and the numerical results obtained by the *RDM6* finite element software.

A schematic of the hyperstatic beam:

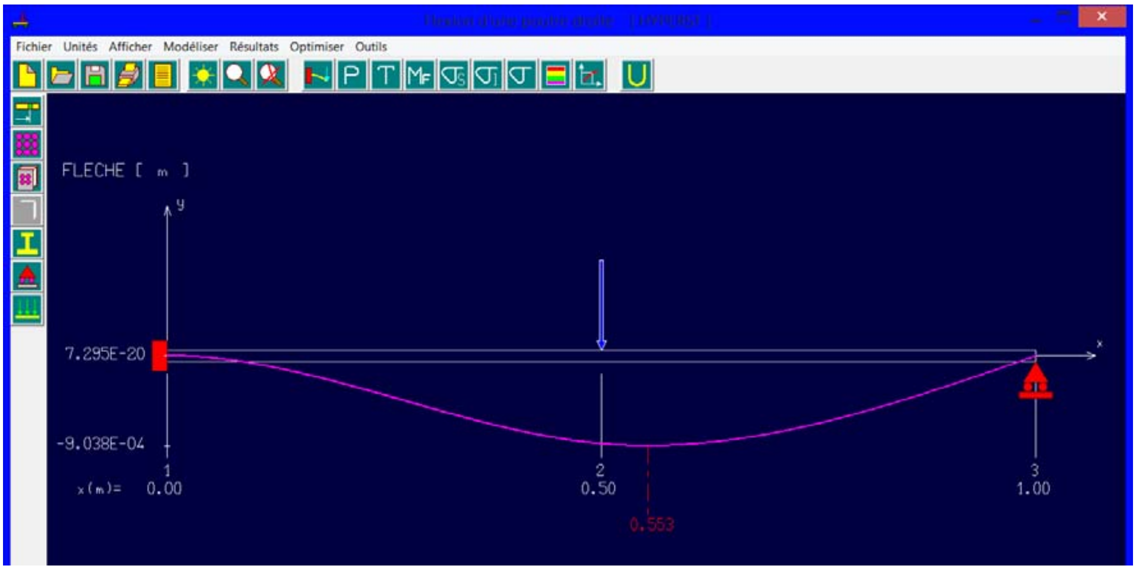


In node A we have $R_A = 68750\,\text{N}$ and in node B we have $R_B = 31250\,\text{N}$.

Variation along the beam of the bending moment obtained by *RDM6*:



Deflection of the beam obtained by *RDM6*:



4.4. Superposition method to resolve the hyperstatic system

We like to calculate the unknowns moment and reactions of the following hyperstatic beam using the superposition method.

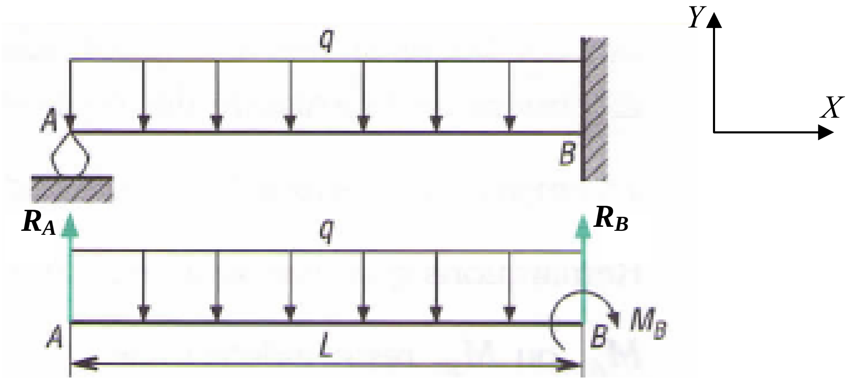


Figure V- 7: Hyperstatic cantilever beam subjected by a uniformly distributed load

Static equilibrium equations:

Using the static equilibrium equations, we have obtained:

$$\sum F_{/Y} = 0 \Rightarrow R_A + R_B - qL = 0 \Rightarrow R_A + R_B = qL$$

$$\sum M_{/B} = 0 \Rightarrow M_B + R_A L - q\frac{L^2}{2} = 0 \Rightarrow M_B + R_A L = q\frac{L^2}{2}$$

Bending moments:

$$0 < x < L \Rightarrow Mf(x) = R_A x - \frac{q}{2}x^2 = \frac{q}{2}x^2 - R_A x$$

The following figure shows the application of the superposition theorem in order to obtain the deflection at the A point when we remove the A support.

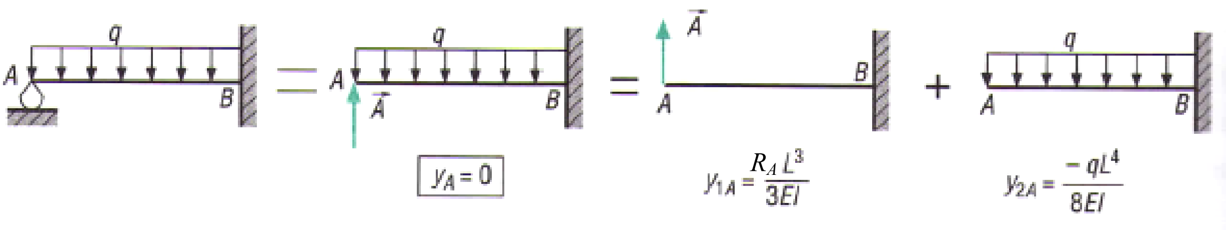


Figure V- 8: Calculation of point A deflection using the superposition procedure

The total deflection of the A point (Figure V- 8) is equal to:

$$y_A = y_{1A} + y_{2A} = \frac{R_A L^3}{3EI_z} - \frac{qL^4}{8EI_z}$$

In reality $y_A = 0$ because we have a support at the A point, then $y_A = 0$

$$y_A = 0 \Rightarrow \frac{R_A L^3}{3EI_z} - \frac{qL^4}{8EI_z} = 0 \Rightarrow R_A = \frac{3qL}{8}$$

Using the previous static equilibrium equations, we can deduce R_B and M_B as:

$$R_A + R_B = qL \Rightarrow R_B = qL - R_A = qL - \frac{3qL}{8}$$

$$\Rightarrow R_B = \frac{5qL}{8}$$

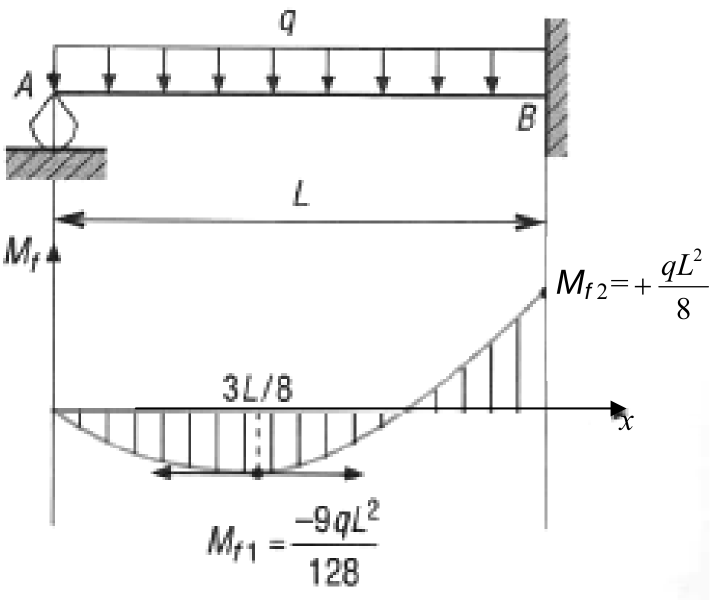
$$M_B + R_A L = q\frac{L^2}{2} \Rightarrow M_B = q\frac{L^2}{2} - R_A L = q\frac{L^2}{2} - \frac{3qL}{8}L$$

$$\Rightarrow M_B = \frac{qL^2}{8}$$

Bending moment diagram:

$$0 < x < L \Rightarrow Mf(x) = R_A x - \frac{q}{2}x^2 \text{ or } Mf(x) = \frac{q}{2}x^2 - R_A x$$

$$\Rightarrow Mf(x) = \frac{q}{2}x^2 - \frac{3qL}{8}x$$



The maximum bending moment Mf_{max} is located at the embedding zone and is equal to $+\frac{qL^2}{8}$.

4.5. Initial parameters method to resolve the hyperstatic system

The beam presented in the following figure is solicited by a non uniformly distributed load, we use the initial parameters method to calculate the reaction R_A and the maximum bending moment Mf_{max}.

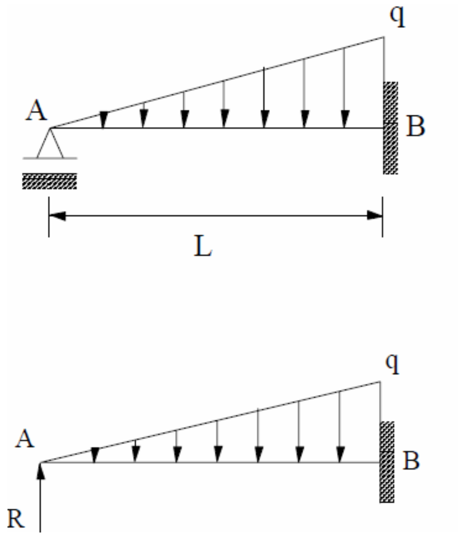


Figure V- 9: Hyperstatic cantilever beam subjected by a non uniformly distributed load

$$q(x) = \frac{q}{L}x$$

Bending moments:

$$0 < x < L \Rightarrow Mf(x) = R_A x - \left[\left(\int_0^x q(x)dx\right) \times \frac{x}{3}\right]$$

$$\Rightarrow Mf(x) = R_A x - \left[\left(\int_0^x \frac{q}{L}xdx\right) \times \frac{x}{3}\right] = R_A x - \frac{q}{6L}x^3$$

The equations of the deformed beam:

$$EI_z\theta(x) = EI_z\theta_0 + \int Mf(x)dx = EI_z\theta_0 + \frac{1}{2}R_A x^2 - \frac{q}{24L}x^4$$

$$EI_z y(x) = EI_z y_0 + \int EI_z\theta(x)dx = EI_z y_0 + EI_z\theta_0 x + \frac{1}{6}R_A x^3 - \frac{q}{120L}x^5$$

The boundary conditions are: $y(L) = \theta(L) = 0$ and $y(0) = y_0 = 0$

$$\begin{cases} \theta(L) = 0 \Rightarrow EI_z\theta_0 + \frac{1}{2}R_A L^2 - \frac{q}{24}L^3 \\ y(L) = 0 \Rightarrow EI_z\theta_0 L + \frac{1}{6}R_A L^3 - \frac{q}{120}L^4 \end{cases} \Rightarrow \begin{cases} R_A = \frac{qL}{10} \\ \theta_0 = \frac{qL^3}{120EI_z} \end{cases}$$

$$\Rightarrow Mf(x) = \frac{qL}{10}x - \frac{q}{6L}x^3$$

$Mf_{\max}$ is located at $x = \frac{L}{\sqrt{5}}$ and is equal to $\frac{qL^2}{15\sqrt{5}}$.

Directed works No. 5 “Solution of hyperstatic systems”

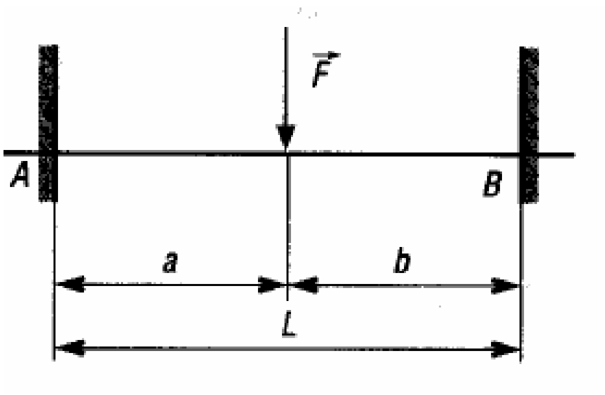
Exercise N°1

The beam presented below is embedded at its both ends A and B, it is subjected to a force *F* applied in its gravity center.

1) Using the integration method and the energetic method, determine the vertical reactions and the embedding moments at the supports A and B.

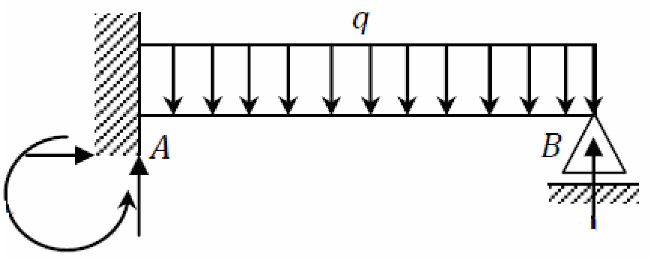
2) Plot the variation of the bending moment along the beam and determine the maximum value of the bending moment.

3) Determine the minimal deflection of the beam.



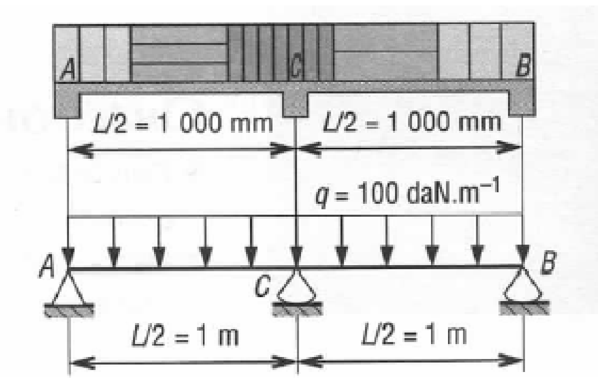
Exercise N°2

Using the superposition method, determine the unknowns reactions and embedding moment of the following hyperstatic beam.



Exercise N°3

Using any method you know of solving hyperstatic systems, calculate the unknowns reactions of the following hyperstatic beam.



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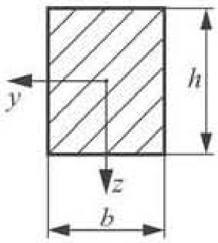
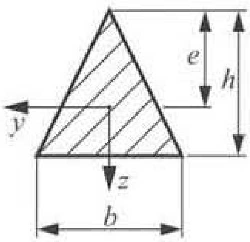
Appendices

Appendices

A1-Area inertia moment [5]

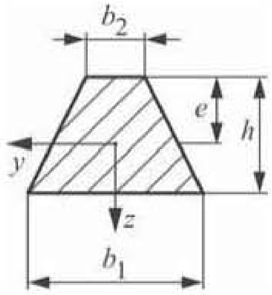


$$I_y = \frac{b \cdot h^3}{12}$$
$$I_z = \frac{h \cdot b^3}{12}$$
$$W_y = \frac{b \cdot h^2}{6}$$
$$W_z = \frac{h \cdot b^2}{6}$$

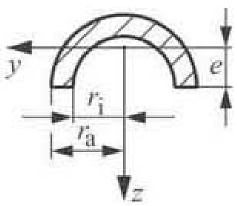


$$e = \frac{2}{3} h$$

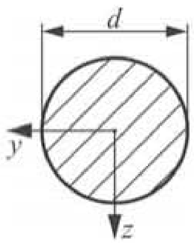
$$I_y = \frac{b \cdot h^3}{36}$$
$$I_z = \frac{h \cdot b^3}{48}$$
$$W_y = \frac{b \cdot h^2}{24}$$
$$W_z = \frac{h \cdot b^2}{24}$$



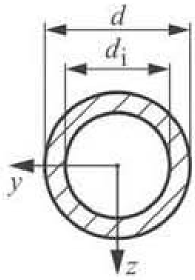
$$I_y = \frac{h^3}{36} \cdot \frac{b_1^{\,2} + 4b_1 \cdot b_2 + b_2^{\,2}}{b_1 + b_2}$$
$$W_y = \frac{h^2}{12} \cdot \frac{b_1^{\,2} + 4b_1 \cdot b_2 + b_2^{\,2}}{2b_1 + b_2}$$
$$e = \frac{h}{3} \cdot \frac{2b_1 + b_2}{b_1 + b_2}$$



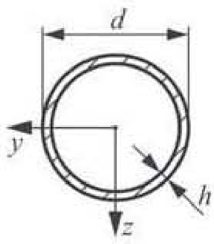
$$I_y = 0{,}1098 \cdot (r_a^{\,4} - r_i^{\,4}) - 0{,}283 \cdot r_a^{\,2} \cdot r_i^{\,2} \cdot \frac{r_a - r_i}{r_a + r_i}$$
$$e = \frac{4}{3\pi} \cdot \frac{r_a^{\,2} + r_a \cdot r_i + r_i^{\,2}}{r_a + r_i}$$



$$I_y = I_z = \frac{\pi \cdot d^4}{64}$$
$$W_y = W_z = \frac{\pi \cdot d^3}{32}$$

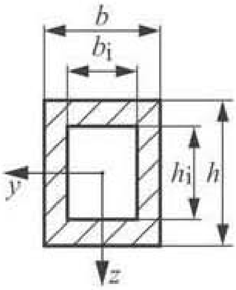


$$I_y = I_z = \frac{\pi}{64} \cdot (d^4 - d_i^{\,4})$$
$$W_y = \frac{\pi}{32d}(d^4 - d_i^{\,4})$$



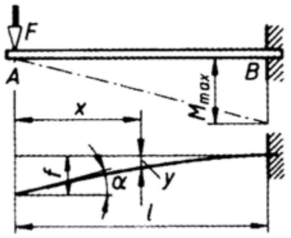
$h << d$

$$I_y = I_z \approx \frac{\pi \cdot d^3 \cdot h}{8}$$
$$W_y = W_z \approx \frac{\pi \cdot d^2 \cdot h}{4}$$



$$I_y = \frac{b \cdot h^3 - b_i \cdot h_i^{\,3}}{12}$$
$$W_y = \frac{b \cdot h^3 - b_i \cdot h_i^{\,3}}{6h}$$

A2-Reactions, moments and deflection [14]

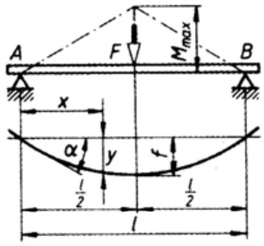


$F_B = F$

$M_{max} = Fl$

$f = \frac{Fl^3}{3EI}$

$y = \frac{Fl^3}{3EI}\left(1 - \frac{3x}{2l} + \frac{x^3}{2l^3}\right)$ $\tan\alpha = \frac{Fl^2}{2EI} = \frac{3f}{2l}$

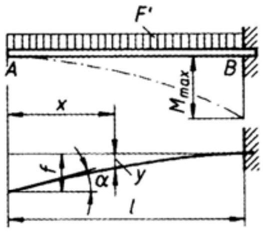


$F_A = F_B = \frac{F}{2}$

$M_{max} = \frac{Fl}{4}$

$f = \frac{Fl^3}{48EI}$

$y = \frac{Fl^2x}{16EI}\left(1 - \frac{4x^2}{3l^2}\right)$ für $x \le \frac{l}{2}$ $\tan\alpha = \frac{Fl^2}{16EI} = \frac{3f}{l}$



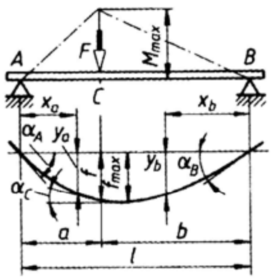
$F_B = F = F'l$

$M_{max} = \frac{F'l^2}{2}$

$f = \frac{F'l^4}{8EI}$

$\tan\alpha = \frac{F'l^3}{6EI} = \frac{4f}{3l}$

$y = \frac{F'l^4}{24EI}\left(\frac{x^4}{l^4} - 4\frac{x}{l} + 3\right)$



$F_A = F\frac{b}{l}$ $F_B = F\frac{a}{l}$

$M_{max} = F\frac{ab}{l}$

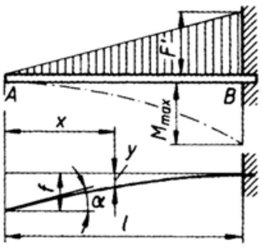
$f = \frac{Fa^2b^2}{EI3l}$

$f_{max} = f\frac{l+a}{3a}\sqrt{\frac{l+a}{3b}}$

$\tan\alpha_A = f\left(\frac{1}{a} + \frac{1}{2b}\right)$ $\tan\alpha_B = f\left(\frac{1}{b} + \frac{1}{2a}\right)$

$y_a = \frac{Fab^2x_a}{6EIl}\left(1 + \frac{l}{b} - \frac{x_a^2}{ab}\right)$ $y_b = \frac{Fa^2bx_b}{6EIl}\left(1 + \frac{l}{a} - \frac{x_b^2}{ab}\right)$

(für $x_a \le a$) (für $x_b \le b$)



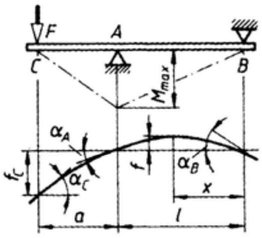
$F_B = F = \frac{F'l}{2}$

$M_{max} = \frac{F'l^2}{6}$

$f = \frac{F'l^4}{30EI}$

$\tan\alpha = \frac{F'l^3}{24EI} = \frac{5f}{4l}$

$y = \frac{F'l^4}{120EI}\left(\frac{x^5}{l^5} - 5\frac{x}{l} + 4\right)$



$F_A = F\left(1 + \frac{a}{l}\right)$ $F_B = F\frac{a}{l}$

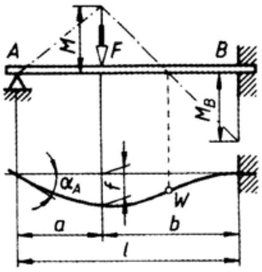
$M_{max} = Fa = M_A$

$f = \frac{Fl^3a}{EI9\sqrt{3}l}$

für $x = 0{,}577l$

$f_C = \frac{Fl^3a^2}{3EIl^2}\left(1 + \frac{a}{l}\right)$

$\tan\alpha_A = \frac{Fal}{3EI}$ $\tan\alpha_B = \frac{Fal}{6EI}$ $\tan\alpha_C = F\frac{a(2l+3a)}{6EI}$



$$F_A = F\frac{b^2}{l^2}\left(1 + \frac{a}{2l}\right)$$

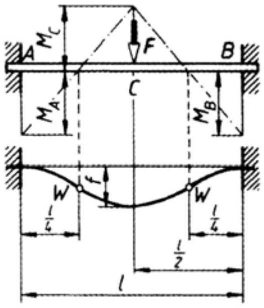
$$F_B = F - F_A$$

$$f = \frac{Fa^2b^3}{4EIl^2}\left(1 + \frac{a}{3l}\right)$$

$$\tan\alpha_A = \frac{Fab^2}{4EIl}$$

$$M = Fa\left[1 + \frac{1}{2}\left(\frac{a}{b}\right)^3 - \frac{3a}{2l}\right]$$

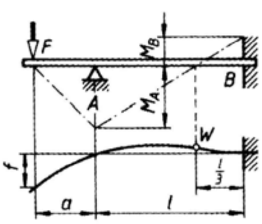
$$M_B = \frac{Fl}{2}\left[\frac{a}{l} - \left(\frac{a}{l}\right)^3\right]$$



$$F_A = F_B = \frac{F}{2}$$

$$M_C = \frac{Fl}{8} = M_A = M_B$$

$$f = \frac{Fl^3}{192EI}$$



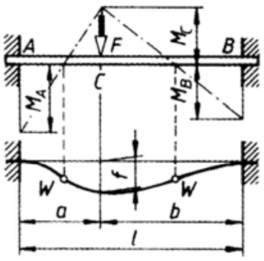
$$F_A = F\left(1 + \frac{3a}{2l}\right)$$

$$F_B = F\frac{3a}{2l}$$

$$M_A = Fa$$

$$M_B = \frac{Fa}{2}$$

$$f = \frac{Fl^3}{EI}\left[\frac{1}{3}\left(\frac{a}{l}\right)^3 + \frac{1}{4}\left(\frac{a}{l}\right)^2\right]$$



$$M_A = Fa\left(\frac{b}{l}\right)^2$$

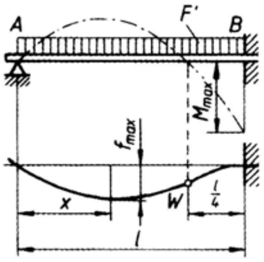
$$M_B = Fb\left(\frac{a}{l}\right)^2$$

$$f = \frac{Fa^3b^3}{3EIl^3}$$

$$M_C = 2Fb\left(\frac{a}{l}\right)^2 \cdot \left(1 - \frac{a}{l}\right)$$

$$F_A = F\left(\frac{b}{l}\right)^2 \cdot \left(3 - 2\frac{b}{l}\right)$$

$$F_B = F\left(\frac{a}{l}\right)^2 \cdot \left(3 - 2\frac{a}{l}\right)$$



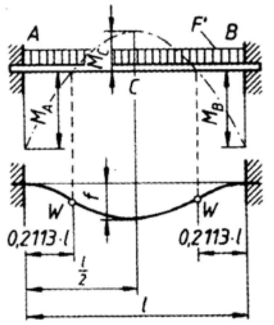
$$F_A = \frac{3}{8}F'l$$

$$F_B = \frac{5}{8}F'l$$

$$M_{max} = \frac{F'l^2}{8}$$

$$f_{max} = \frac{F'l^4}{185EI}$$

für $x = 0{,}4215l$

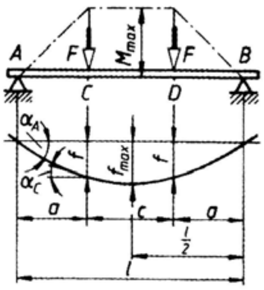


$$F_A = F_B = \frac{F'l}{2}$$

$$M_C = \frac{F'l^2}{24}$$

$$M_A = M_B = \frac{F'l^2}{12} = M_{max}$$

$$f = \frac{F'l^4}{384EI}$$

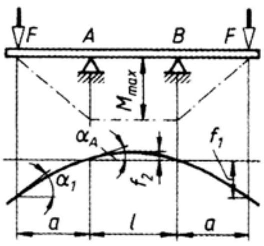


$$F_A = F_B = F$$

$$M_{max} = Fa$$

$$f = \frac{Fl^3a^2}{2EIl^2}\left(1 - \frac{4a}{3l}\right) \qquad \tan\alpha_A = \frac{Fa\,(a + c)}{2EI}$$

$$f_{max} = \frac{Fl^3a}{8EIl}\left(1 - \frac{4a^2}{3l^2}\right) \qquad \tan\alpha_C = \tan\alpha_D = \frac{Fac}{2EI}$$



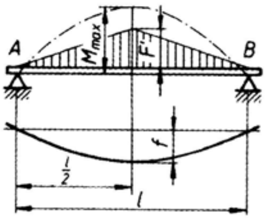
$$F_A = F_B = F$$

$$M_{max} = Fa$$

$$f_1 = \frac{Fa^2}{EI}\left(\frac{a}{3} + \frac{l}{2}\right)$$

$$f_2 = \frac{Fal^2}{8EI}$$

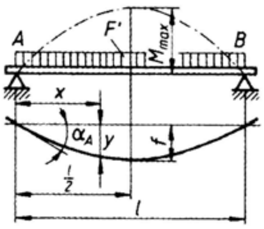
$$\tan\alpha_1 = \frac{Fa\,(l + a)}{2EI} \qquad \tan\alpha_A = \frac{Fal}{2EI}$$



$$F_A = F_B = \frac{F'l}{4}$$

$$M_{max} = \frac{F'l^2}{12}$$

$$f = \frac{F'l^4}{120EI}$$



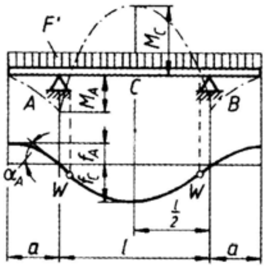
$$F_A = F_B = \frac{F'l}{2}$$

$$M_{max} = \frac{F'l^2}{8}$$

$$f = \frac{5}{384} \cdot \frac{F'l^4}{EI}$$

$$\tan\alpha_A = \frac{F'l^3}{24EI} = \frac{16f}{5l}$$

$$y = \frac{F'l^3x}{24EI}\left(1 - \frac{x}{l}\right)\left(1 + \frac{x}{l} - \frac{x^2}{l^2}\right)$$



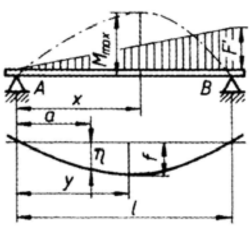
$$F_A = F_B = F'\left(\frac{l}{2} + a\right)$$

$$M_A = \frac{F'a^2}{2}$$

$$M_C = \frac{F'l^2}{2}\left[\frac{1}{4} - \left(\frac{a}{l}\right)^2\right]$$

$$\tan\alpha_A = \frac{F'l^3}{4EI}\left[\frac{1}{6} - \left(\frac{a}{l}\right)^2\right] \qquad f_A = \frac{F'l^4}{4EI}\left[\frac{a}{6l} - \left(\frac{a}{l}\right)^3 - \frac{1}{2}\left(\frac{a}{l}\right)^4\right]$$

$$f_C = \frac{F'l^4}{16EI}\left[\frac{5}{24} - \left(\frac{a}{2}\right)^2\right]$$



$$F_A = \frac{F'l}{6} \qquad F_B = \frac{F'l}{3}$$

$$M_{max} = 0{,}064F'l^2$$

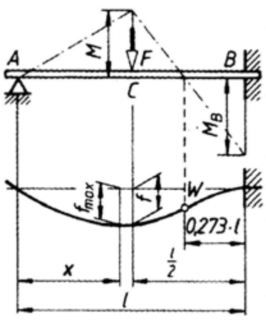
bei $x = 0{,}5774l$

$$f = \frac{F'l^4}{153{,}4EI}$$

bei $y = 0{,}5193l$

$$\eta = \frac{F'l^3a}{360EI}\left(1 - \frac{a^2}{l^2}\right)\left(7 - 3\,\frac{a^2}{l^2}\right)$$

F in Stabmitte



$$F_A = \frac{5}{16}F \qquad F_B = \frac{11}{16}F$$

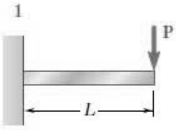
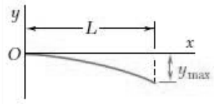
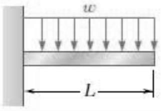
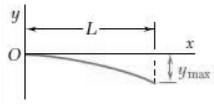
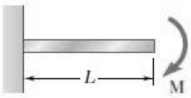
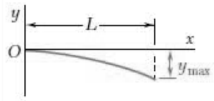
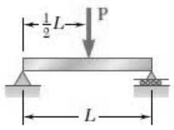
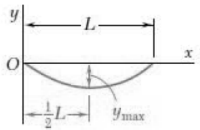
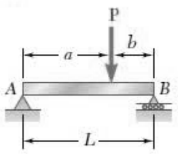
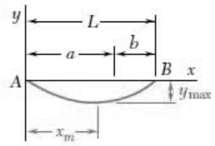
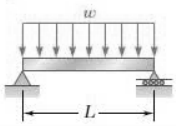
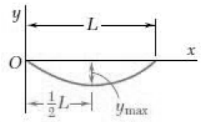
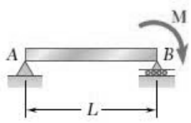
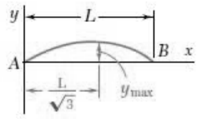
$$M = \frac{5}{32}Fl$$

$$M_B = \frac{3}{16}Fl$$

$$f = \frac{7Fl^3}{768EI}$$

$$f_{max} = \frac{Fl^3}{48\sqrt{5}EI} \text{ bei } x = 0{,}447l$$

A3-Beam deflections and slopes [12]

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \leq \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2 - b^2}{3}}$	$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2 - a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$

A4-Materials properties [12]

		Ultimate Strength			Yield Strength[3]					
Material	Density kg/m^3	Tension, MPa	Compression,[2] MPa	Shear, MPa	Tension, MPa	Shear, MPa	Modulus of Elasticity, GPa	Modulus of Rigidity, GPa	Coefficient of Thermal Expansion, $10^{-6}/°C$	Ductility, Percent Elongation in 50 mm
Steel										
Structural (ASTM-A36)	7860	400			250	145	200	77.2	11.7	21
High-strength-low-alloy										
ASTM-A709 Grade 345	7860	450			345		200	77.2	11.7	21
ASTM-A913 Grade 450	7860	550			450		200	77.2	11.7	17
ASTM-A992 Grade 345	7860	450			345		200	77.2	11.7	21
Quenched & tempered										
ASTM-A709 Grade 690	7860	760			690		200	77.2	11.7	18
Stainless, AISI 302										
Cold-rolled	7920	860			520		190	75	17.3	12
Annealed	7920	655			260	150	190	75	17.3	50
Reinforcing Steel										
Medium strength	7860	480			275		200	77	11.7	
High strength	7860	620			415		200	77	11.7	
Cast Iron										
Gray Cast Iron										
4.5% C, ASTM A-48	7200	170	655	240			69	28	12.1	0.5
Malleable Cast Iron										
2% C, 1% Si,										
ASTM A-47	7300	345	620	330	230		165	65	12.1	10
Aluminum										
Alloy 1100-H14										
(99% Al)	2710	110		70	95	55	70	26	23.6	9
Alloy 2014-T6	2800	455		275	400	230	75	27	23.0	13
Alloy-2024-T4	2800	470		280	325		73		23.2	19
Alloy-5456-H116	2630	315		185	230	130	72		23.9	16
Alloy 6061-T6	2710	260		165	240	140	70	26	23.6	17
Alloy 7075-T6	2800	570		330	500		72	28	23.6	11
Copper										
Oxygen-free copper										
(99.9% Cu)										
Annealed	8910	220		150	70		120	44	16.9	45
Hard-drawn	8910	390		200	265		120	44	16.9	4
Yellow-Brass										
(65% Cu, 35% Zn)										
Cold-rolled	8470	510		300	410	250	105	39	20.9	8
Annealed	8470	320		220	100	60	105	39	20.9	65
Red Brass										
(85% Cu, 15% Zn)										
Cold-rolled	8740	585		320	435		120	44	18.7	3
Annealed	8740	270		210	70		120	44	18.7	48
Tin bronze	8800	310			145		95		18.0	30
(88 Cu, 8Sn, 4Zn)										
Manganese bronze	8360	655			330		105		21.6	20
(63 Cu, 25 Zn, 6 Al, 3 Mn, 3 Fe)										
Aluminum bronze	8330	620	900		275		110	42	16.2	6
(81 Cu, 4 Ni, 4 Fe, 11 Al)										

Material	Density kg/m^3	Ultimate Strength: Tension, MPa	Ultimate Strength: Compression,[2] MPa	Ultimate Strength: Shear, MPa	Yield Strength[3]: Tension, MPa	Yield Strength[3]: Shear, MPa	Modulus of Elasticity, GPa	Modulus of Rigidity, GPa	Coefficient of Thermal Expansion, $10^{-6}/°C$	Ductility, Percent Elongation in 50 mm
Magnesium Alloys										
Alloy AZ80 (Forging)	1800	345		160	250		45	16	25.2	6
Alloy AZ31 (Extrusion)	1770	255		130	200		45	16	25.2	12
Titanium										
Alloy (6% Al, 4% V)	4730	900			830		115		9.5	10
Monel Alloy 400(Ni-Cu)										
Cold-worked	8830	675			585	345	180		13.9	22
Annealed	8830	550			220	125	180		13.9	46
Cupronickel (90% Cu, 10% Ni)										
Annealed	8940	365			110		140	52	17.1	35
Cold-worked	8940	585			545		140	52	17.1	3
Timber, air dry										
Douglas fir	470	100	50	7.6			13	0.7	Varies	
Spruce, Sitka	415	60	39	7.6			10	0.5	3.0 to 4.5	
Shortleaf pine	500		50	9.7			12			
Western white pine	390		34	7.0			10			
Ponderosa pine	415	55	36	7.6			9			
White oak	690		51	13.8			12			
Red oak	660		47	12.4			12			
Western hemlock	440	90	50	10.0			11			
Shagbark hickory	720		63	16.5			15			
Redwood	415	65	42	6.2			9			
Concrete										
Medium strength	2320		28				25		9.9	
High strength	2320		40				30		9.9	
Plastics										
Nylon, type 6/6, (molding compound)	1140	75	95		45		2.8		144	50
Polycarbonate	1200	65	85		35		2.4		122	110
Polyester, PBT (thermoplastic)	1340	55	75		55		2.4		135	150
Polyester elastomer	1200	45		40			0.2			500
Polystyrene	1030	55	90		55		3.1		125	2
Vinyl, rigid PVC	1440	40	70		45		3.1		135	40
Rubber	910	15							162	600
Granite (Avg. values)	2770	20	240	35			70	4	7.2	
Marble (Avg. values)	2770	15	125	28			55	3	10.8	
Sandstone (Avg. values)	2300	7	85	14			40	2	9.0	
Glass, 98% silica	2190		50				65	4.1	80	