Moroccan J. of Pure and Appl. Anal. (MJPAA)
Volume 6(2), 2020, Pages 218–230
ISSN: Online 2351-8227 - Print 2605-6364
DOI 10.2478/mjpaa-2020-0017

Boundary Value Problem for Caputo–Fabrizio Random Fractional Differential Equations

Fouzia Bekada[1], Saïd Abbas[2], and Mouffak Benchohra[3]

Abstract. This article deals with some existence of random solutions and Ulam stability results for a class of Caputo-Fabrizio random fractional differential equations with boundary conditions in Banach spaces. Our results are based on the fixed point theory and random operators. Two illustrative examples are presented in the last section.

Mathematics Subject Classification (2010). 26A33
Key words and phrases. Fractional differential equation, Caputo–Fabrizio integral of fractional order, Caputo–Fabrizio fractional derivative, Random solution, Ulam stability, fixed point.

1. Introduction

Fractional calculus and fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [1, 25]. In recent years, several works and development of fractional differential equation and

Received date 25 June 2020 - Accepted: 20 August 2020.

The Author(s): [1]*Laboratory of Mathematics, Tahar Moulay University of Saïda, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria*
e-mail: bekadafouzia@gmail.com
[2]*Department of Mathematics, Tahar Moulay University of Saïda, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria*
e-mail: abbasmsaid@yahoo.fr
[2]*Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria*
e-mail: benchohra@yahoo.com .

inclusions are cited to the monographs [1, 3, 4, 5, 15, 17, 24, 27, 29, 30], the papers [2, 6, 7, 18, 20, 29] and the references therein.

There are different definitions of fractional derivatives. The popular derivatives of fractional order we mention Riemann-Liouville, Caputo, Hadamard, and Hilfer. For example; the Caputo fractional derivative of order $0 < \alpha < 1$ of a function $g \in L^1[0,T];\ T > 0$, is given by

$$ {}^cD_0^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} g'(s)ds. $$

Caputo and Fabrizio developed and proposed a new version of fractional derivative by changing the Kernel $(t-s)^{-\alpha}$ by the function $(t,s) \mapsto \exp(\frac{(-\alpha(t-s))}{(1-\alpha)})$ and $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$. For more details; see [11].

The question of stability for functional differential equations was introduced by Ulam and Hyers. Thereafter; this type of stability is called the Ulam-Hyers stability [16, 23]. In 1978, Rassias provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables of stability for a functional equation arises when we replace the functional equation by an inequality. For more details; see the monographs [5, 9, 12, 13], the papers [8, 14, 19, 21, 22, 23, 26, 27, 28], and the references therein

In [19], the authors used the Laplace transform method and established the existence and HyersUlam stability of initial value problems of linear Caputo–Fabrizio fractional differential equations. In the present paper we investigate the following class of random Caputo–Fabrizio fractional differential equation

$$ ({}^{CF}D_0^\alpha u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T],\ w \in \Omega, \tag{1.1} $$

with the boundary conditions

$$ au(0,w) + bu(T,w) = c(w);\ w \in \Omega, \tag{1.2} $$

where $T > 0,\ f : I \times E \times \Omega \to E$ is a given function, $a, b \in \mathbb{R},\ , c : \Omega \to E$, with $a + b \neq 0$, ${}^{CF}D_0^\alpha$ is the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$, and Ω is the sample space in a probability space (Ω, F), and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

2. Preliminaries

Let $\mathcal{C} := C(I,E)$ be the Banach space of all continuous functions from I into E with the norm

$$ \|u\|_\infty = \sup\{\|u(t)\| : t \in I\}. $$

By $L^1(I,E)$ we denote the Banach space of measurable function $u : I \to E$ with are Bochner integrable, equipped with the norm

$$ \|u\|_{L^1} = \int_0^T \|u(t)\|dt. $$

Definition 2.1. [11] *The Caputo-Fabrizio fractional integral of order $0 < \alpha < 1$ for a function $h \in L^1(I)$ is defined by*

$$^{CF}I^{\alpha}h(\tau) = \frac{2(1-\alpha)}{M(\alpha)(2-\alpha)}h(\tau) + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_0^{\tau} h(x)dx, \;\; \tau \geq 0,$$

where $M(\alpha)$ is normalization constant depending on α.

Definition 2.2. [11] *The Caputo-Fabrizio fractional derivative for a function $h \in C^1(I)$ of order $0 < \alpha < 1$, is defined by*

$$^{CF}D^{\alpha}h(\tau) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^{\tau} \exp(-\frac{\alpha}{1-\alpha}(\tau - x))h'(x)dx; \; \tau \in I.$$

Note that $(^{CF}D^{\alpha})(h) = 0$ if and only if h is a constant function.

Lemma 2.1. *Let $h \in L^1(I, E)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} (^{CF}D_0^{\alpha}u)(t) = h(t); \;\; t \in I := [0, T] \\ au(0) + bu(T) = c, \end{cases} \tag{2.1}$$

where $a, b \in \mathbb{R}$, $c \in E$ with $a + b \neq 0$, if and only if, u satisfies the following integral equation

$$u(t) = C_0 + a_{\alpha}h(t) + b_{\alpha}\int_0^t h(s)ds + \frac{bb_{\alpha}}{a+b}\int_0^T h(s)ds, \tag{2.2}$$

where

$$a_{\alpha} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}, \;\; b_{\alpha} = \frac{2\alpha}{(2-\alpha)M(\alpha)},$$
$$C_0 = \frac{1}{a+b}[c - ba_{\alpha}(h(T) - h(0))] - a_{\alpha}h(0).$$

Proof. Suppose that u satisfies (2.1). From Proposition 1 in [11]; the equation $(^{CF}D_0^{\alpha}u)(t) = h(t)$ implies that

$$u(t) - u(0) = a_{\alpha}(h(t) - h(0)) + b_{\alpha}\int_0^t h(s)ds.$$

Thus,

$$u(T) = u(0) + a_{\alpha}(h(T) - h(0)) + b_{\alpha}\int_0^T h(s)ds.$$

From the mixed boundary conditions $au(0) + bu(T) = c$, we get

$$au(0) + b(u(0) + a_{\alpha}(h(T) - h(0)) + b_{\alpha}\int_0^T h(s)ds) = c.$$

Hence,

$$u(0) = \frac{c - b(a_{\alpha}(h(T) - h(0)) - b_{\alpha}\int_0^T h(s)ds)}{a+b}.$$

So; we get (2.2).
Conversely, if u satisfies (2.2), then $(^{CF}D_0^{\alpha}u)(t) = h(t)$; for $t \in I := [0, T]$, and $au(0) + bu(T) = c$.

From the above lemma, we can conclude with the following lemma:

Lemma 2.2. *A function u is a random solution of problem (1.1)-(1.2), if and only if u satisfies the following integral equation:*

$$u(t,w) = C_0(w) + a_\alpha f(t,u(t,w),w)$$
$$+b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds,$$

where

$$C_0(w) = \frac{1}{a+b}[c(w) - ba_\alpha(f(T,u(T,w),w) - f(0,u(0,w),w))] - a_\alpha f(0,u(0,w),w).$$

Let β_E be the $\sigma-$algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \subset \Omega : v(w) \subset B\} \subset A.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.3. *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \subset \beta_E$, one has*

$$T^{-1}(B) = \{(w,v) \subset \Omega \times E : T(w,v) \subset B\} \subset A \times \beta_E$$

where $A \times \beta_E$ is the direct product of the $\sigma-$algebras A and β_E those defined in Ω and E respectively.

Lemma 2.3. *Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \subset E$, and $T(w, \cdot)$ is continuous for all $w \subset \Omega$. Then the map $(w,v) \to T(w,v)$ is jointly measurable.*

Definition 2.4. *A function $f : I \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

- *(i) The map $(t,w) \to f(t,u,w)$ is jointly measurable for all $u \subset E$, and*
- *(ii) The map $u \to f(t,u,w)$ is continuous for almost all $t \in I$ and $w \subset \Omega$.*

Definition 2.5. *$T : \Omega \times E \to E$ be a mapping. then T is called a random operator if $T(w,u)$ is measurable in w for all $u \subset E$ and it id expressed as $T(w)u = T(w,u)$. In this case we also say that $T(w)$ is random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous)in u for all $w \subset \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh.*

Definition 2.6. *Let $P(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $P(Y)$. A mapping $T : \{(w,u) : w \subset \Omega, y \subset C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e for all closed $A \subset Y$, $\{w \subset \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $u \subset Y, \{w \subset \Omega : u \subset C(w), T(w,u) \subset D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T , a mapping $u : \Omega \to Y$ is called random (stochastic) fixed point of T if for P-almost all $w \subset \Omega, u(w) \subset C(w)$ and $T(w)u(w) = u(w)$ and for all open $D \subset Y, \{w \subset \Omega : u(w) \subset D\}$ is measurable.*

Now, we consider the Ulam stability for the problem (1.1)-(1.2). Let $\epsilon > 0$ and $\Phi : I \times \Omega \to \mathbb{R}_+$ be a measurable function. We consider the following inequalities

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \epsilon;\ t \in I,\ w \in \Omega. \tag{2.3}$$

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \Phi(t,w);\ t \in I,\ w \in \Omega. \tag{2.4}$$

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \epsilon\Phi(t,w);\ t \in I,\ w \in \Omega. \tag{2.5}$$

Definition 2.7. [5] *The problem(1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u(\cdot,w) \in C(I)$ of the inequality (2.3), there exists a solution $v() \in C(I)$ of (1.1)-(1.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_f;\ t \in I.$$

Definition 2.8. [5] *The problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u(w) \in C(I)$ of the inequality (2.3), there exists a solution $v \in C(I)$ of (1.1)-(1.2) with*

$$\|u(t) - v(t)\| \leq c_f(\epsilon);\ t \in I.$$

Definition 2.9. [5] *The problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u(w) \in C(I)$ of the inequality (2.5), there exists a solution $v \in C(I)$ of (1.1)-(1.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_{f,\phi}\phi(t,w);\ t \in I.$$

Definition 2.10. [5] *The problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each solution $u \in C(I)$ of the inequality (2.4), there exists a solution $v(w) \in C(I)$ of (1.1)-(1.2) with*

$$\|u(t) - v(t)\| \leq c_{f,\phi}\phi(t,w);\ t \in I.$$

Remark 2.1. *A function $u(\cdot,w) \in \mathcal{C}$ is a solution of the inequality (2.4) if and only if there exist a function $g(\cdot,w) \in \mathcal{C}$ (which depend on u) such that*

$$\|g(t,w)\| \leq \Phi(t,w),$$

and

$$({}^{CF}D_0^{\alpha}u)(t,w) = f(t,u(t,w)) + g(t,w);\ \textit{for}\ t \in I,\ \textit{and}\ w \in \Omega.$$

In the sequel, we will use the following fixed point Theorem:

Theorem 2.1. [10] *Let X be a nonempty, closed convex bounded subset of the separable Banach space X and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

3. Existence of solutions

Definition 3.1. *By a random solution of problem (1.1)-(1.2), we mean a function $u \in C$ that satisfies the equation*

$$u(t,w) = C_0(w) + a_\alpha f(t,u(t,w),w)$$
$$+b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds,$$

where

$$C_0(w) = \frac{1}{a+b}[c(w) - ba_\alpha(f(T,u(T,w),w) - f(0,u(0,w),w))] - a_\alpha f(0,u(0,w),w).$$

The following hypotheses will be used in the sequel:

(H_1)**:** The function f is random Carathéodory.

(H_2)**:** There exist measurable and bounded functions $p_i : \Omega \to C(I,[0,\infty));\ i = 1,2$ such that

$$\|f(t,u,w)\| \leq p_1(t,w) + p_2(t,w)\|u\|;$$

for all $u \subset E$ and $t \in I$ with

$$p_i^*(w) = \sup_{t\in I} p_i(t,w);\ i = 1,2,\ w \in \Omega.$$

Now, we prove an existence result for the problem (1.1)-(1.2) based on Itoh's fixed point theorem.

Theorem 3.1. *Assume that the hypotheses $(H_1) - (H_2)$ hold. If*

$$\left(a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right)p_2^*(w) < 1, \tag{3.1}$$

then the problem (1.1)-(1.2) has at least one random solution defined on I.

Proof. From Lemma 2.2 for any $w \in \Omega$ and each $t \in I$, the problem (1.1)-(1.2) is equivalent to the operator equation $(Nw)u = u$, where $N : \Omega \times \mathcal{C} \to \mathcal{C}$ be the operator defined by $(Nu)(t,w) = C_0(w) + a_\alpha f(t,u(t,w),w)$

$$+ b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds. \tag{3.2}$$

Since the function f is absolutely continuous for all $w \in \Omega$ and $t \in I$, then u is a random solution for the problem (1.1)-(1.2) if and only if $u = (Nu)(t,w)$. Set

$$R(w) > \frac{\|C_0(w)\| + \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right]p_1^*(w)}{1 - \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right]p_2^*(w)}\quad w \in \Omega. \tag{3.3}$$

Define the ball

$$B_R = B(0,R(w)) := \{u \in \mathcal{C} : \|u\| \leq R(w)\}.$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
\|(Nu)(t,w)\| &\leq \|C_0(w)\| + \|a_\alpha f(t,u(t,w),w)\| \\
&+ \left\|b_\alpha \int_0^t f(s,u(s,w),w)ds\right\| + \left\|\frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\right\| \\
&\leq \|C_0(w)\| + a_\alpha\|f(t,u(t,w),w)\| \\
&+ b_\alpha \int_0^t \|f(s,u(s,w),w)\|ds + \frac{bb_\alpha}{a+b}\int_0^T \|f(s,u(s,w),w)\|ds \\
&\leq \|C_0(w)\| + \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right](p_1^*(w) + p_2^*(w)R(w)) \\
&\leq R(w).
\end{aligned}$$

This proves that $N(w)$ transforms the ball B_R into itself. We shall prove in three steps that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 2.1.

Step 1. $N(w)$ is a random operator.
Since $f(t,u,w)$ is random Carathéodory, the map $w \longrightarrow f(t,u,w)$ is measurable in view Definition 2.6 and further the integral is a limit of a finite sum of measurable functions therefore the map

$$\begin{aligned}
w &\mapsto C_0(w) + a_\alpha f(t,u(t,w),w) \\
&+ b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds,
\end{aligned}$$

is measurable. As a result, $N(w)$ is a random operator.

Step 2. $N(w)$ is continuous and bounded.
Let u_n be a sequence such that $u_n \to U$ in $\mathcal{C}$. Then, for each $t \in I$ we have

$$\begin{aligned}
\|(Nu_n)(t,w) - (Nu)(t,w)\| &\leq \|a_\alpha(f(t,u(t,w),w) - f(t,u_n(t,w),w))\| \\
&+ \left\|b_\alpha \int_0^t (f(t,u(t,w),w) - f(t,u_n(t,w),w))ds\right\| \\
&+ \left\|\frac{bb_\alpha}{a+b}\int_0^T (f(t,u(t,w),w) - f(t,u_n(t,w),w))\right\| \\
&\leq a_\alpha\|f(t,u(t,w),w) - f(t,u_n(t,w),w)\| \\
&+ b_\alpha \int_0^t \|f(t,u(t,w),w) - f(t,u_n(t,w),w)\|ds \\
&+ \frac{bb_\alpha}{a+b}\int_0^T \|f(t,u(t,w),w) - f(t,u_n(t,w),w)\|ds.
\end{aligned}$$

Since f is Carathéodory, then by the Lebesgue dominated convergence theorem, we get

$$\|(Nu_n)(\cdot,w)) - (Nu)(\cdot,w)\|_\infty \to 0 \; as \; n \to \infty.$$

Since $N(w)$ is a continuous random operator with stochastic domain. We can conclude that $N(w)B_R \subset B_R$ is bounded.

Step 3. $N(w)B_R$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{aligned}
\|(Nu)(t_2,w) &- (Nu)(t_1,w)\| \leq \Big\| a_\alpha f(t_2,u(t_2,w),w) + b_\alpha \textstyle\int_0^{t_2} f(s,u(s,w),w)ds \\
&+ \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds - a_\alpha f(t_1,u(t_1,w),w) \\
&- b_\alpha \int_0^{t_1} f(s,u(s,w),w)ds - \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds \Big\| \\
&\leq a_\alpha \|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)\| \\
&+ b_\alpha \int_{t_1}^{t_2} \|f(s,u(s,w),w)ds\| \\
&\leq a_\alpha \|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)\| \\
&+ b_\alpha (t_2 - t_1)(p_1^*(w) + p_2^*(w)R(w)) \\
&\to 0 \ as \ t_2 \to t_1.
\end{aligned}$$

As a consequence of the above steps and the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 2.1, the operator equation $Nu(w) = u$ has a random solution.

4. Ulam stability results

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.1)-(1.2). The following hypotheses will be used in the sequel.

(H_3)**:** $\Phi(\cdot,w) \in L^1(\mathbb{R}_+)$, and there exists a measurable and bounded function $q : \Omega \to C(I,[0,\infty))$; such that

$$(1 + \|u-v\|)\|f(t,u(t,w),w) - f(t,v(t,w),w)\| \leq q(t,w)\Phi(t,w)\|u-v\|;$$

for all $u,v \in E$ and each $t \in I$, with

$$q^*(w) = \sup_{t\in I} q(t,w); \ w \in \Omega.$$

(H_4)**:** There exists a constant $\lambda_\Phi > 0$, such that for any $w \in \Omega$, and each $t \in I$ we have

$$\int_0^T \Phi(t,w)dt \leq \lambda_\Phi \Phi(t,w).$$

Remark 4.1. *From (H_3), for any $w \in \Omega$, and each $t \in I$, and $u \in E$, we have that*

$$\|f(t,u,w)\| \leq \|f(t,0,w)\| + q(t,w)\Phi(t,w)\|u\|.$$

So, (H_3) implies (H_2), with $p_1(t,w) = \|f(t,0,w)\|$, and $p_2(t,w) = q(t,w)\Phi(t,w)$,

Lemma 4.1. *If $u \in \mathcal{C}$ is a solution of the inequality (2.4) then u is a solution of the following integral inequality*

$$\|u(t,w) - C_0(w) - a_\alpha f(s,u(s,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds$$

$$-\frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\| \leq \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w);\ t\in I;\ w\in\Omega. \tag{4.1}$$

Proof. By Remark 2.1; for any $w\in\Omega$ and each $t\in I$, we have

$$\begin{aligned} u(t,w) &= C_0(w) + a_\alpha[f(s,u(s,w),w)+g(s,w)] \\ &+ b_\alpha \int_0^t [f(s,u(s,w),w)+g(s,w)]ds \\ &+ \frac{bb_\alpha}{a+b}\int_0^T [f(s,u(s,w),w)+g(s,w)]ds. \end{aligned}$$

Thus, we get

$$\begin{aligned} \|u(t,w) &- C_0(w) - a_\alpha f(s,u(s,w),w) - b_\alpha\int_0^t f(s,u(s,w),w)ds \\ &- \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\| \\ &\leq a_\alpha\|g(s,w)\| + b_\alpha\int_0^t \|g(s,w)\|ds + \frac{bb_\alpha}{a+b}\int_0^T \|g(s,w)\|ds \\ &\leq \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi\frac{bb_\alpha}{a+b}\right)\Phi(t,w). \end{aligned}$$

Theorem 4.1. *Assume that the hypotheses $(H_1),(H_3),(H_4)$ and the condition (3.1) hold. Then the problem (1.1)-(1.2) has at least one solution on I and it is generalized Ulam-Hyers-Rassias stable.*

Proof. From Remark 4.1, there exists a random solution v of the random problem (1.1)-(1.2). That is

$$\begin{aligned} v(t,w) &= C_0(w) + a_\alpha f(t,v(t,w),w) \\ &+ b_\alpha\int_0^t f(s,v(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,v(s,w),w)ds. \end{aligned}$$

Let u be a solution of the inequality (2.4), then from Lemma 4.1, for any $w\in\Omega$, and each $t\in I$, we have

$$\begin{aligned} \|u(t,w) &- C_0(w) + a_\alpha f(t,u(t,w),w) - b_\alpha\int_0^t f(s,u(s,w),w)ds \\ &- \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\bigg\| \leq \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi\frac{bb_\alpha}{a+b}\right)\Phi(t,w). \end{aligned}$$

Then, for any $w\in\Omega$, and each $t\in I$, we obtain

$$\begin{aligned} \|u(t,w)-v(t,w)\| &\leq \|u(t,w) - C_0(w) - a_\alpha f(t,u(t,w),w) - b_\alpha\int_0^t f(s,u(s,w),w)ds \\ &- \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds + a_\alpha f(t,u(t,w),w) \end{aligned}$$

$$\begin{aligned}
&+ \quad b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds \\
&- \quad a_\alpha f(t,v(t,w),w) - b_\alpha \int_0^t f(s,v(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b}\int_0^T f(s,v(s,w),w)ds\|.
\end{aligned}$$

This implies that,

$$\begin{aligned}
\|u(t,w)-v(t,w)\| \quad &\leq \quad \|u(t,w) - C_0(w) - a_\alpha f(t,u(t,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\| \\
&+ \quad \|a_\alpha f(t,u(t,w),w) + b_\alpha \int_0^t f(s,u(s,w),w)ds - a_\alpha f(t,v(t,w),w) \\
&- \quad b_\alpha \int_0^t f(s,v(s,w),w)ds - \frac{bb_\alpha}{a+b}\int_0^T f(s,v(s,w),w)ds\| \\
&\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha \|f(t,u(t,w),w) - f(t,v(t,w),w)\| \\
&+ \quad b_\alpha \int_0^t \|f(s,u(s,w),w) - f(s,v(s,w),w)\|ds \\
&+ \quad \frac{bb_\alpha}{a+b}\int_0^T \|f(s,u(s,w),w) - f(s,v(s,w),w)\|ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|u(t,w)-v(t,w)\| \quad &\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha q^*(w)\Phi(t,w)\frac{\|u(t,w)-v(t,w)\|}{1+\|u(t,w)-v(t,w)\|} \\
&+ \quad b_\alpha \int_0^t q^*(w)\Phi(t,w)\frac{\|u(s,w)-v(s,w)\|}{1+\|u(s,w)-v(s,w)\|}ds \\
&+ \quad \frac{bb_\alpha}{a+b}\int_0^T q^*(w)\Phi(t,w)\frac{\|u(s,w)-v(s,w)\|}{1+\|u(s,w)-v(s,w)\|}ds \\
&\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha q^*(w)\Phi(t,w) + b_\alpha q^*(w)\int_0^t \Phi(t,w)ds \\
&+ \quad \frac{bb_\alpha q^{*(w)}}{a+b}\int_0^T \Phi(t,w)ds.
\end{aligned}$$

Hence, from (H_4), we get

$$\begin{aligned}\|u(t,w)-v(t,w)\| &\leq \left(a_\alpha+\lambda_\Phi b_\alpha+\lambda_\Phi\frac{bb_\alpha}{a+b}\right)\Phi(t,w)+a_\alpha q^*(w)\Phi(t,w)\\ &+ \left(b_\alpha q^*(w)+\frac{bb_\alpha q^*(w)}{a+b}\right)\int_0^T\Phi(s,w)ds\\ &\leq \left(a_\alpha+\lambda_\Phi b_\alpha+\lambda_\Phi\frac{bb_\alpha}{a+b}\right)(1+q^*(w))\Phi(t,w)\\ &:= c_{f,\Phi}\Phi(t,w).\end{aligned}$$

This conclude that our problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable.

5. Examples

Let $\Omega=(-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$, and

$$E=l^1=\left\{u=(u_1,u_2,\dots,u_n,\dots),\sum_{n=1}^{\infty}|u_n|<\infty\right\}$$

be the Banach space with the norm

$$\|u\|_E=\sum_{n=1}^{\infty}|u_n|.$$

Example 1. Consider the Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^\alpha u_n)(t,w)=\frac{cw^2(2^{-n}+u_n(t,w))}{\exp(t+3)(1+w^2+|u_n(t,w)|)};\ t\in[0,1],\ w\in\Omega, \tag{5.1}$$

with the boundary conditions

$$u_n(0,w)+u_n(1,w)=\frac{1}{1+w^2};\ w\in\Omega. \tag{5.2}$$

Set $0<c<\frac{2}{2a_\alpha+3b_\alpha}$, and

$$f(t,u(t,w),w)=\frac{cw^2(2^{-n}+u_n(t,w)))}{\exp(t+3)(1+w^2+|u(t,w)|)};\ t\in[0,1],\ w\in\Omega.$$

The hypothesis (H_2) is satisfied with

$$p_1(t,w)=p_2(t,w)=\frac{cw^2}{1+w^2}e^{-t},$$

and then

$$p_1^*(w)=p_2^*(w)=c.$$

The condition (3.1) is satisfied. Indeed;

$$\left(a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right) p_2^*(w) = c\left(a_\alpha + \frac{3b_\alpha}{2}\right) < 1,$$

Consequently, Theorem 3.1 implies that the problem (5.1)-(5.2) has at least one random solution defined on $[0,1]$.

Example 2. Consider now the Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^\alpha u_n)(t,w) = \frac{cw^2 2^{-n}}{\exp(t+3)(1+w^2+|u_n(t,w)|)};\ t\in[0,1],\ w\in\Omega, \tag{5.3}$$

with the boundary conditions

$$u_n(0,w) + u_n(1,w) = \frac{w}{1+w^2};\ w\in\Omega. \tag{5.4}$$

Set

$$f(t,u(t,w),w) = \frac{cw^2 2^{-n}}{\exp(t+3)(1+w^2+|u(t,w)|)};\ t\in[0,1],\ w\in\Omega.$$

The hypothesis (H_3) is satisfied with

$$q(t,w) = \frac{cw^2}{1+w^2} \text{ and } \Phi(t) = e^{-t}.$$

The condition (3.1) is satisfied with a good choice of the constant c.
Also; the hypotheses (H_4) is satisfied with $\lambda_\Phi = e-1$. Indeed;

$$\int_0^T \Phi(t,w)dt = \int_0^T e^{-t}dt = 1-e^{-1} \le \lambda_\Phi e^{-t} = \lambda_\Phi\Phi(t,w);\ t\in[0,1].$$

Consequently, Theorem 4.1 implies that the problem (5.3)-(5.4) has at least one random solution and it is generalized-Ulam-Hyers-Rassias stable.

References

[1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[2] S. Abbas, M. Benchohra and H. Gorine, Caputo-Hadamard fractional differential equations with four-point boundary conditions, *Commun. Appl. Nonlinear Anal.* **26** (3) (2019), 68-79.

[3] S. Abbas, M. Benchohra, N. Hamidi and J. Henderson, Caputo-Hadamard fractional differential equations in Banach spaces, *Frac. Calc. Appl. Anal.* **21** (4) (2018), 1027-1045.

[4] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[5] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[6] S. Abbas, M. Benchohra and A. Petrusel, Ulam stabilities for the Darboux problem for partial fractional differential inclusions via Picard Operators, *Electron. J. Qual. Theory Differ. Equ.* **51** (2014), 1-13.

[7] S. Abbas, M. Benchohra, A. Petrusel, Ulam stability for Hilfer type fractional differential inclusions via the weakly Picard operator theory, *Frac. Calc. Appl. Anal.* **20** (2) (2017), 384-398.

[8] S. Abbas, M. Benchohra and S. Sivasundaram, Ulam stability for partial fractional differential inclusions with multiple delay and impulses via Picard operators, *Nonlinear Stud.* **20** (4) (2013), 623-641.

[9] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhuser, 1998.
[10] S. Itoh, Random fixed point theorems with applications to random differential equations in Banach spaces, *J. Math. Anal. Appl* **67** (1979), 261-273.
[11] J. Losada and J.J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**(2) (2015), 87-92.
[12] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis.*, Hadronic Press, Palm Harbor, 2001.
[13] S.-M. Jung, A fixed point approach to the stability of a Volterra integral equation, *Fixed Point Theory Appl.* 2007 (2007), Article ID 57064, 9 pages.
[14] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis.*, Springer, New York, 2011.
[15] A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38** (6) (2001) 1191-1204.
[16] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27** (1941), 222-224.
[17] A.A. Kilbas, H.M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V.,Amsterdam, 2006.
[18] V. Lakshmikantham, and J. Vasundhara Devi, Theory of fractional differential equations in a Banach space, *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.
[19] K. Liu , M. Fečkan , D. O'Regan and J.R. Wang, Hyers-Ulam Stability and existence of solutions for differential equations with Caputo-Fabrizio fractional derivative, *Mathematics*, **7** 333 (2019), 1-14.
[20] H.Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal.* **4** (1980), 985-999.
[21] M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
[22] I. A. Rus, Ulam stability of operatorial equations, *Fixed Point Theory* **10** (2009), 305-320.
[23] I. A. Rus, Ulam stability of ordinary diferential equations, *Studia Univ. Babes-Bolyai, Math.* LIV (4)(2009), 125-133.
[24] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.
[25] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[26] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative. *E. J. Qual. Theory Diff. Equ.* (63) (2011) 1-10.
[27] J. Wang, L. Lv, Y. Zhou, New concepts and results in stability of fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.* 17 (2012), 2530-2538.
[28] W. Wei, X. Li, X. Li, New stability results for fractional integral equation, *Comput. Math. Appl.* 64 (2012), 3468-3476.
[29] M. Yang, Q. Wang, Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions. *Fract. Calc. Appl. Anal.* **20** (2017), 679-705.
[30] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.