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WELL-POSEDNESS AND GENERAL ENERGY DECAY OF SOLUTION FOR TRANSMISSION PROBLEM WITH WEAKLY NONLINEAR DISSIPATIVE

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In this paper, we consider a transmission problem in a bounded domain with a nonlinear dissipation in the first equation. Under suitable assumptions on the weight of the damping, we show the existence and uniqueness of solution by the Faedo–Galerkin method. Also we prove general stability estimates using some properties of convex functions and Lyaponov functional.

1. Introduction

In this paper, we consider a nonlinear transmission problem

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu\varpi(u_t(x,t)) = 0 & (x,t) \in \Omega \times \mathbb{R}^+, \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0 & (x,t) \in [L_1, L_2] \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $0 < L_1 < L_2 < L_3 < \infty$, $\Omega =]0, L_1[\cup]L_2, L_3[$ and a, b, μ are positive constants. This system is supplemented with the following boundary and transmission conditions

$$\begin{aligned} u(0,t) &= u(L_3,t) = 0 \\ u(L_i,t) &= v(L_i,t), \qquad i = 1,2 \\ a u_x(L_i,t) &= b v_x(L_i,t), \qquad i = 1,2, \end{aligned} \tag{1.2}$$

and initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \\ v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in]L_1, L_2[. \end{aligned} \tag{1.3}$$

When $g(u_t(x,t)) = u_t(x,t)$ system (1.1)–(1.3) has been investigated in [3], for $\Omega = [0, L_1]$. The authors showed the well-posedness and exponential stability of the total energy. Ma and Oquendo in [12] considered transmission problem involving two Euler–Bernoulli equations modeling the vibrations of a composite beam. By using just one boundary damping term in the boundary, they showed the global existence and decay property of solution. Marzocchi et al. in [13] investigated a 1-D semilinear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero, no matter how small the damping subdomain is. A similar result has sheen shown in [15], where a transmission problem in thermoelasticity of type III has been investigated. See also Marzocchi et al. [14] for a multidimensional linear thermoelastic transmission

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problem. To obtain global solution of problem (1.1)–(1.3), we use the Galerkin approximation scheme (see Lions [10]) together with the energy estimate method. To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [6], Lasiecka and Doundykov [9], Lasiecka and Tataru [8] and used by Liu and Zuazua [11] and Alabau-Boussouira [1] and Zennir et al. [4], [5], [16]. Our purpose in this paper is organized as follows. In Section 2, we give some preliminaries. While Sections 3 and 4 are devoted to the global existence, uniqueness and general decay of solutions, respectively. Then main results are in Theorems 3.1 and 4.1.

2. Preliminaries

First we recall and make use the following assumptions on the function ϖ. We assume that the function $\varpi \in C^1(\mathbb{R}, \mathbb{R})$ is a nondecreasing function such that there exist a positive constants $\varepsilon, c_1, c_2 > 0$ and a convex increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(]0, +\infty[)$ satisfying

$$\begin{gathered} G(0) = 0 \text{ and } G \text{ is linear on } [0, \varepsilon] \text{ or} \\ G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0, \varepsilon] \text{ such that} \\ |\varpi(s)| \le c_2|s| \quad \text{if } |s| > \varepsilon \\ s^2 + \varpi^2(s) \le G^{-1}(s\varpi(s)) \quad \text{if } |s| \le \varepsilon \\ |\varpi'(s)| \le \tau. \end{gathered} \tag{2.4}$$

We first state some lemmas which will be needed later.

Lemma 1 (Sobolev–Poincaré's inequality). *Let q be a number such that $2 \le q \le +\infty$ $(n = 1, 2)$ or $2 \le q \le 2n/(n-2)$ $(n \ge 3)$. Then there exists a constant $C_s = C((0, 1), q)$ such that*

$$\|u\|_q \le C_s \|\nabla u\|_2, \quad \textit{for all } u \in H_0^1(\Omega).$$

Remark. Let us denote by ϕ^* the conjugate function of the differentiable convex function ϕ, i. e.,

$$\phi^*(s) = \sup_{t \in \mathbb{R}_+} (st - \phi(t)).$$

Then ϕ^* is the Legendre transform of ϕ, which is given by (see Arnold [2, pages 61–62])

$$\phi^*(s) = s(\phi')^{-1}(s) - \phi((\phi')^{-1}(s)), \quad \text{if } s \in]0, \phi'(r)],$$

and ϕ^* satisfies the generalized Young inequality

$$ST \le \phi^*(S) + \phi(T), \quad \text{if } S \in]0, \phi'(r)], T \in]0, r]. \tag{2.5}$$

3. Well-posedness of problem

In this section, we prove the existence and uniqueness of a global solution of system (1.1)–(1.3) by using the Faedo–Galerkin method.

Theorem 3.1. *Suppose that* $\{u^0, v^0\} \in H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)$, $\{u^1, v^1\} \in H_0^1(\Omega) \times H_0^1(L_1, L_2)$ *and assumption* (2.4) *holds. Then* (1.1)–(1.3) *admits a unique global solution*

$$\begin{aligned} \{u, v\} &\in L^\infty(0, T, H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)),\\ \{u_t, v_t\} &\in L^\infty(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2)),\\ \{u_{tt}, v_{tt}\} &\in L^\infty(0, T, L^2(\Omega) \times L^2(L_1, L_2)). \end{aligned}$$

Proof of Theorem 3.1. We follow a number of steps to complete the proof.

Step 1. Approximate solutions. Let $\{\varphi^i, \psi^i\}$, $i = 1, 2, \ldots$ be a basis of $H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)$. Let us consider the Galerkin approximation

$$\{u^m(t), v^m(t)\} = \sum_{i=1}^{m} h^{im}(t)\{\varphi^i, \psi^i\},$$

where u^m and v^m satisfy

$$(u_{tt}^m, \varphi^i) + a(u_x^m, \varphi_x^i) + \mu(\varpi(u_t^m), \varphi^i) + (v_{tt}^m, \psi^i) + b(v_x^m, \psi_x^i) = 0, \tag{3.6}$$

where $i = 1, 2, \ldots$, with initial data

$$\begin{aligned} \{u^m(0), v^m(0)\} = \{u_0^m, v_0^m\} &\to \{u^0, v^0\} \quad \text{in } H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2),\\ \{u_t^m(0), v_t^m(0)\} = \{u_1^m, v_1^m\} &\to \{u^1, v^1\} \quad \text{in } H_0^1(\Omega) \times H_0^1(L_1, L_2). \end{aligned} \tag{3.7}$$

Standard results about ordinary differential equations guarantee that there exists only one solution of this system on some interval $[0, T_m[$. The priori estimate that follow imply that in fact $T_m = +\infty$.

Step 2. A priori estimates.

The first estimate. Multiplying (3.6) by h_t^{im} and summing over i, we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |u_t^m|^2\,dx + a\int_\Omega |u_x^m|^2\,dx + \int_{L_1}^{L_2} |v_t^m|^2\,dx + b\int_{L_1}^{L_2} |v_x^m|^2\,dx\right\} + \mu\int_\Omega u_t^m \varpi(u_t^m)\,dx = 0. \tag{3.8}$$

Integrating in $[0, t]$, $t < t_m$ and using (3.7), we have

$$\begin{aligned} \int_\Omega |u_t^m|^2\,dx + a\int_\Omega |u_x^m|^2\,dx + \int_{L_1}^{L_2} |v_t^m|^2\,dx + b\int_{L_1}^{L_2} |v_x^m|^2\,dx + 2\int_0^t \int_\Omega u_t^m(s)\varpi(u_t^m(s))\,dx\,ds&\\ \le \int_\Omega |u_1^m|^2\,dx + a\int_\Omega |u_0^m|^2\,dx + \int_{L_1}^{L_2} |v_1^m|^2\,dx + b\int_{L_1}^{L_2} |v_0^m|^2\,dx&\\ \le C_1.& \end{aligned} \tag{3.9}$$

For some C_1 independent of m.

Thus we deduce that

$$\begin{aligned} &\{u^m, v^m\} && \text{is bounded in } L^\infty(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2))\\ &\{u_t^m, v_t^m\} && \text{is bounded in } L^\infty(0, T, L^2(\Omega) \times L^2(L_1, L_2))\\ &u_t^m \varpi(u_t^m) && \text{is bounded in } L^1(\Omega \times (0, T)). \end{aligned} \tag{3.10}$$

The second estimate. First, we estimate $u_{tt}^m(0)$ and $v_{tt}^m(0)$ taking $t=0$ in (3.6), we obtain

$$(u_{tt}^m(0),\varphi^i)-a(u_{xx}^m(0),\varphi^i)+\mu(\varpi(u_t^m(0)),\varphi^i)=0,$$

and

$$(v_{tt}^m(0),\psi^i)-b(v_{xx}^m(0),\psi^i)=0,$$

multiplying by h_{tt}^{im} and summing over i from 1 to m,

$$(u_{tt}^m(0),u_{tt}^m(0))-a(u_{xx}^m(0),u_{tt}^m(0))+\mu(\varpi(u_t^m(0)),u_{tt}^m(0))=0,$$

and

$$(v_{tt}^m(0),v_{tt}^m(0))-b(v_{xx}^m(0),v_{tt}^m(0))=0.$$

Using Hölder's inequality and (3.7), yield

$$\begin{aligned}(3.11)\quad \Big(\int_\Omega |u_{tt}^m(0)|^2\,dx\Big)^{1/2}&+\Big(\int_{L_1}^{L_2}|v_{tt}^m(0)|^2\,dx\Big)^{1/2}\\ &\le a\Big(\int_\Omega |u_{xx}^m(0)|^2\,dx\Big)^{1/2}+\mu\Big(\int_\Omega \varpi^2(u_1^m)\,dx\Big)^{1/2}+b\Big(\int_{L_1}^{L_2}|v_{xx}^m(0)|^2\,dx\Big)^{1/2}\\ &\le C_2,\end{aligned}$$

where C_2 is a positive constant independent of m.

The third estimate. Now, differentiating (3.6) with respect to t

$$(u_{ttt}^m,\varphi^i)-a(u_{txx}^m,\varphi^i)+\mu(u_{tt}^m\varpi'(u_t^m),\varphi^i)+(v_{ttt}^m,\psi^i)-b(v_{txx}^m,\psi^i)=0.$$

Multiplying by h_{tt}^{mi} and summing over i from 1 to m implies

$$(3.12)\quad \frac{1}{2}\frac{d}{dt}\Big[\int_\Omega|u_{tt}^m|^2\,dx+a\int_\Omega|u_{xt}^m|^2\,dx+\int_{L_1}^{L_2}|v_{tt}^m|^2\,dx+b\int_{L_1}^{L_2}|v_{xt}^m|^2\,dx\Big]+\mu\int_\Omega(u_{tt}^m)^2\varpi'(u_t^m)\,dx=0.$$

Integrating (3.12) over $(0,t)$, using (3.7) and (3.11), we get

$$\begin{aligned}\int_\Omega|u_{tt}^m(t)|^2\,dx&+a\int_\Omega|u_{xt}^m(t)|^2\,dx+\int_{L_1}^{L_2}|v_{tt}^m(t)|^2\,dx\\ &+b\int_{L_1}^{L_2}|v_{xt}^m(t)|^2\,dx+2\mu\int_0^t\int_\Omega(u_{tt}^m(s))^2\varpi'(u_t^m(s))\,dx\,dt\\ &=\int_\Omega|u_{tt}^m(0)|^2\,dx+a\int_\Omega|u_{xt}^m(0)|^2\,dx+\int_{L_1}^{L_2}|v_{tt}^m(0)|^2\,dx+b\int_{L_1}^{L_2}|v_{xt}^m(0)|^2\,dx\\ &\le C_3,\end{aligned}$$

where C_3 is a positive constant independent of m.

Therefore, we conclude that

$$(3.13)\quad \begin{aligned}&\{u_t^m,v_t^m\}\quad \text{is bounded in } L^\infty(0,T,H_0^1(\Omega)\times H_0^1(L_1,L_2))\\ &\{u_{tt}^m,v_{tt}^m\}\quad \text{is bounded in } L^\infty(0,T,L^2(\Omega)\times L^2(L_1,L_2)),\end{aligned}$$

By (3.13) we deduce that

$$\{u_t^m, v_t^m\} \text{ is bounded in } L^2(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2)).$$

Applying Rellich compactness theorem given in [10], we deduce that

$$\{u_t^m, v_t^m\} \text{ is bounded in } L^2(0, T, L^2(\Omega) \times L^2(L_1, L_2)). \tag{3.14}$$

The fourth estimate. Replacing φ^i and ψ^i by $-u_{xx}^m$ and $-v_{xx}^m$ in (3.6), multiplying the result by h_t^{im}, summing over i from 1 to m, implies

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u_{tx}^m|^2\,dx + a\int_\Omega |u_{xx}^m|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m|^2\,dx + b\int_{L_1}^{L_2} |v_{xx}^m|^2\,dx\right] + \mu\int_\Omega (u_{tx}^m)^2\varpi'(u_t^m)\,dx = 0. \tag{3.15}$$

Integrating (3.15) over $(0, t)$ and using (3.7), we have

$$\begin{aligned}\int_\Omega |u_{tx}^m(t)|^2\,dx &+ a\int_\Omega |u_{xx}^m(t)|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m(t)|^2\,dx \\ &+ b\int_{L_1}^{L_2} |v_{xx}^m(t)|^2\,dx + \mu\int_0^t\int_\Omega (u_{tx}^m(s))^2\varpi'(u_t^m(s))\,dx\,ds \\ &= \int_\Omega |u_{tx}^m(0)|^2\,dx + a\int_\Omega |u_{xx}^m(0)|^2\,dx + \int_{L_1}^{L_2} |v_{tx}^m(0)|^2\,dx + b\int_{L_1}^{L_2} |v_{xx}^m(0)|^2\,dx \\ &\le C_4,\end{aligned} \tag{3.16}$$

where C_4 is a positive constant independent of m.

We conclude that

$$\begin{aligned}&\{u^m, v^m\} \quad \text{is bounded in } L^\infty(0, T, H^2(\Omega) \times H^2(L_1, L_2)) \\ &\{u_t^m, v_t^m\} \quad \text{is bounded in } L^\infty(0, T, H_0^1(\Omega) \times H_0^1(L_1, L_2)),\end{aligned} \tag{3.17}$$

Step 3. Passing to the limit. Applying Dunford–Petti's theorem, we conclude from (3.10), (3.13) and (3.17), after replacing the sequences $\{u^m, v^m\}$ by subsequence if necessary, that

$$\{u^m, v^m\} \rightharpoonup^* \{u, v\}, \quad \text{in } L^\infty(0, T; H^2(\Omega) \times H^2(L_1, L_2) \cap H_0^1(\Omega) \times H_0^1(L_1, L_2)) \tag{3.18}$$

$$\{u_t^m, v_t^m\} \rightharpoonup^* \{u_t, v_t\}, \quad \text{in } L^\infty(0, T; H_0^1(\Omega) \times H_0^1(L_1, L_2)) \tag{3.19}$$

$$\{u_{tt}^m, v_{tt}^m\} \rightharpoonup^* \{u_{tt}^m, v_{tt}^m\}, \quad \text{in } L^\infty(0, T; L^2(\Omega) \times L^2(L_1, L_2)) \tag{3.20}$$

$$\varpi(u_t^m) \rightharpoonup^* \chi, \quad \text{in } L^2(Q), \tag{3.21}$$

where $Q = (0, T) \times \Omega$.

It follows from (3.18) and (3.20), that for each fixed $w_1 \in L^2([0, T] \times L^2(\Omega))$

$$\int_0^T\int_\Omega (u_{tt}^m(x, t) - au_{xx}^m(x, t))w_1\,dx\,dt \to \int_0^T\int_\Omega (u_{tt}(x, t) - au_{xx}(x, t))w_1\,dx\,dt,$$

and $w_2 \in L^2([0,T] \times L^2(L_1, L_2))$

$$\int_0^T \int_{L_1}^{L_2} (v_{tt}^m(x,t) - bv_{xx}^m(x,t)) w_2 \, dx \, dt \to \int_0^T \int_{L_1}^{L_2} (v_{tt}(x,t) - bv_{xx}(x,t)) w_2 \, dx \, dt.$$

By (3.14), (3.19) and the injection of H_0^1 in L^2 is compact, we have

$$u_t^m \to u_t, \quad \text{in } L^2(Q). \tag{3.22}$$

Therefore

$$u_t^m \to u_t, \quad \text{almost everywhere in } Q. \tag{3.23}$$

It remains to show that

$$\int_Q \varpi(u_t^m) v \, dx \, dt \to \int_Q \varpi(u_t) v \, dx \, dt,$$

in the following lemma.

Lemma 2. *For each $T > 0$, $\varpi(u_t) \in L^1(Q)$, $\|\varpi(u_t)\|_{L^1(Q)} \le k$, where k is a constant independent of t and $\varpi(u_t^m) \to \varpi(u_t)$ in $L^1(Q)$.*

Proof. We claim that

$$\varpi(u_t) \in L^1(Q).$$

Indeed, since ϖ is continuous, we deduce from (3.23) that

$$\begin{aligned} \varpi(u_t^m) &\to \varpi(u_t) \quad && \text{almost everywhere in } Q, \\ u_t^m \varpi(u_t^m) &\to u_t \varpi(u_t) \quad && \text{almost everywhere in } Q. \end{aligned} \tag{3.24}$$

Hence, by (3.10) and Fatou's lemma, we have

$$\int_Q u_t(x,t) \varpi(u_t(x,t)) \, dx \, dt \le K_1, \quad \text{for } T > 0. \tag{3.25}$$

Now, we can estimate $\int_Q |\varpi(u_t(x,t))| \, dx \, dt$.

By the Cauchy–Schwarz inequality, we have

$$\int_0^T \int_\Omega |\varpi(u_t(x,t))| \, dx \, dt \le c|Q|^{1/2} \left(\int_0^T \int_\Omega |\varpi(u_t(x,t))|^2 \, dx \, dt \right)^{1/2}.$$

Using (2.4) and (3.25), we obtain, for $T > 0$,

$$\begin{aligned} \int_0^T \int_\Omega |\varpi(u_t(x,t))|^2 \, dx \, dt &\le \int_0^T \int_{|u_t| > \varepsilon} u_t \varpi(u_t) \, dx \, dt + \int_0^T \int_{|u_t| \le \varepsilon} G^{-1}(u_t \varpi(u_t)) \, dx \, dt \\ &\le c \int_0^T \int_\Omega u_t \varpi(u_t) \, dx \, dt + c G^{-1}\left(2 \int_Q u_t \varpi(u_t) \, dx \, dt \right) \\ &\le c \int_0^T \int_\Omega u_t \varpi(u_t) \, dx \, dt + c' G^*(1) + c'' \int_0^T \int_\Omega u_t \varpi(u_t) \, dx \, dt \\ &\le c K_1 + c' G^*(1), \end{aligned}$$

Then

$$\int_0^T \int_Q |\varpi(u_t(x,t))|\, dx\, d \leq K, \quad \text{for } T > 0.$$

Let $E \subset \Omega \times [0, T]$ and set

$$E_1 = \left\{ (x,t) \in E : |\varpi(u_t^m(x,t))| \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If

$$M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |\varpi(s)| \geq r\},$$

we have

$$\int_E |\varpi(u_t^m)|\, dx\, dt \leq c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |u_t^m \varpi(u_t^m)|\, dx\, dt.$$

By applying (3.10) we deduce that

$$\sup_m \int_E \varpi(u_t^m)\, dx\, dt \to 0, \quad \text{when } |E| \to 0.$$

From Vitali's convergence theorem we deduce that

$$\varpi(u_t^m) \to \varpi(u_t) \quad \text{in } L^1(Q).$$

This completes the proof of Lemma 2. □

Then (3.21) implies that

$$\varpi(u_t^m) \rightharpoonup^* \varpi(u_t), \quad \text{in } L^2(Q).$$

We deduce, for all $w_1 \in L^2([0,T] \times L^2(\Omega)$, that

$$\int_0^T \int_\Omega \varpi(u_t^m) w_1\, dx\, dt \to \int_0^T \int_\Omega \varpi(u_t) w_1\, dx\, dt.$$

Finally we have shown that, for all $w_1 \in L^2([0,T] \times L^2(\Omega))$,

$$\int_0^T \int_\Omega (u_{tt}(x,t) - a u_{xx}(x,t) - \mu \varpi(u_t)) w_1\, dx\, dt = 0.$$

Uniqueness. Let u_1, u_2 be two solutions of $(1.1)_1$ and v_1, v_2 be two solutions of $(1.1)_2$ with the same initial data. It is straightforward to see that $z = u_1 - u_2$ and $w = v_1 - v_2$ satisfies

$$\int_\Omega z_t^2(x,t)\, dx + a\int_\Omega z_x^2(x,t)\, dx + \int_{L_1}^{L_2} w_t^2(x,t)\, dx + b\int_{L_1}^{L_2} w_x^2\, dx + \mu \int_0^t \int_\Omega z_t(s)\varpi(z_t(s))\, dx\, ds = 0.$$

Using the monotonicity of ϖ hence we conclude that

$$\int_\Omega z_t^2(x,t)\, dx + a\int_\Omega z_x^2(x,t)\, dx + \int_{L_1}^{L_2} w_t^2(x,t)\, dx + b\int_{L_1}^{L_2} w_x^2\, dx \leq 0,$$

which implies $z = 0$ and $w = 0$. This finishes the proof of Theorem 3.1. □

4. Asymptotic behavior

In this section, we state and prove our stability result for the energy of solution for system (1.1)–(1.3), using the multiplied techniques.

The energies of first and second order associated with system (1.1)–(1.3) are defined as follows

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \frac{a}{2}\int_\Omega u_x^2(x,t)\,dx, \tag{4.26}$$

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)\,dx. \tag{4.27}$$

The total energy is defined as

$$E(t) = E_1(t) + E_2(t). \tag{4.28}$$

Our decay result reads as follows.

Theorem 4.1. *Let (u, v) be the solution of (1.1)–(1.3). Assume that (2.4) holds and*

$$\frac{b}{a} < \frac{L_3 + L_1 - L_2}{2(L_2 - L_1)}. \tag{4.29}$$

Then there exist positive constants k_1, k_2, k_3 and ε_0 such that the solution of the problem (1.1)–(1.3) satisfies

$$E(t) \le k_3 G_1^{-1}(k_1 t + k_2), \quad \forall t \in \mathbb{R}_+, \tag{4.30}$$

where

$$G_1(t) = \int_t^1 \frac{1}{sG_2'(\varepsilon_0 s)}ds, \quad G_2(t) = tG'(\varepsilon_0 t). \tag{4.31}$$

Here G_1 is strictly decreasing and convex on $]0, 1]$, *with* $\lim_{t\to 0} G_1(t) = +\infty$.

For the proof of Theorem 4.1 we use the following lemmas.

Lemma 3. *The total energy $E(t)$ satisfies*

$$E'(t) = -\mu \int_\Omega u_t(x,t)\varpi(u_t(x,t))\,dx \le 0. \tag{4.32}$$

Proof. Multiplying equation $(1.1)_1$ by u_t and integrating in Ω, we have

$$\int_\Omega u_t(x,t)u_{tt}(x,t)\,dx - a\int_\Omega u_t(x,t)u_{xx}(x,t)\,dx = -\mu\int_\Omega u_t(x,t)\varpi(u_t(x,t))\,dx,$$

which integrated by parts leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega [u_t^2(x,t) + au_x^2(x,t)]\,dx &\\ = -\mu\int_\Omega u_t(x,t)\varpi(u_t(x,t))\,dx &- a(u_x(L_1,t)u_t(L_1,t) - u_x(0,t)u_t(0,t))\\ &- a(u_x(L_3,t)u_t(L_3,t) - u_x(L_2,t)u_t(L_2,t)).\end{aligned} \tag{4.33}$$

Multiplying equation (1.1)$_2$ by v_t and performing an integration in (L_1, L_2), we get

$$\int_{L_1}^{L_2} v_t(x,t)v_{tt}(x,t)\,dx - b\int_{L_1}^{L_2} v_t(x,t)v_{xx}(x,t)\,dx = 0.$$

After integrating by parts we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2}[v_t^2(x,t)+bv_x^2(x,t)]\,dx = -b(v_t(L_2,t)v_x(L_2,t)-v_t(L_1,t)v_x(L_1,t)). \tag{4.34}$$

By Section 3 and (4.34), using the transmission conditions (1.2) we conclude

$$\frac{d}{dt}E(t) = -\mu\int_\Omega u_t(x,t)\varpi(u_t(x,t))\,dx.$$

This completes the proof. □

Lemma 4. *Let* (u, v) *be the solution of* (1.1)–(1.3). *Then, the functional*

$$J(t) = \int_\Omega u(x,t)u_t(x,t)dx + \int_{L_1}^{L_2} v(x,t)v_t(x,t)dx, \tag{4.35}$$

satisfies, for any $\delta > 0$, *the estimate*

$$\frac{d}{dt}J(t) \le \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)\,dx - (a-\delta C_s)\int_\Omega u_x^2(x,t)\,dx + b\int_{L_1}^{L_2} v_x^2(x,t)\,dx + C(\delta)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx. \tag{4.36}$$

Proof. Taking the derivative of $J(t)$ with respect to t and using (1.1), we find

$$\frac{d}{dt}J(t) = \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)\,dx - a\int_\Omega u_x^2(x,t)\,dx - b\int_{L_1}^{L_2} v_x^2(x,t)\,dx - \mu\int_\Omega u(x,t)\varpi(u_t(x,t))\,dx + [auu_x]_{\partial\Omega} + [bvv_x]_{L_1}^{L_2}.$$

Using the boundary conditions (1.2), we have

$$\begin{aligned}[auu_x]_{\partial\Omega} + [bvv_x]_{L_1}^{L_2} &= a\{u(L_1,t)u_x(L_1,t) - u(0,t)u_x(0,t)\}\\ &\quad + a\{u(L_3,t)u_x(L_3,t) - u(L_2,t)u_x(L_2,t)\}\\ &\quad + b\{v(L_2,t)v_x(L_2,t) - v(L_1,t)v_x(L_1,t)\}\\ &= 0.\end{aligned}$$

Applying Young and Poincaré's inequalities, we have

$$\mu\int_\Omega u(x,t)\varpi(u_t(x,t))dx \le \delta C_s\int_\Omega u_x^2(x,t)\,dx + C(\delta)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx,$$

where δ is a positive constant. We arrive at

$$\begin{aligned}\frac{d}{dt}J(t) &\leq \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)dx - (a-\delta C_s)\int_\Omega u_x^2(x,t)\,dx \\ &\qquad -b\int_{L_1}^{L_2} v_x^2(x,t)dx + C(\delta)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx \\ &\leq \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)dx - (a-\delta C_s)\int_\Omega u_x^2(x,t)\,dx \\ &\qquad +b\int_{L_1}^{L_2} v_x^2(x,t)dx + C(\delta)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx.\end{aligned}$$

This completes the proof. □

Now, inspired by [13], we introduce the functional

$$q(x) = \begin{cases} x - L_1/2 & x\in[0,L_1], \\ x-(L_2+L_3)/2 & x\in[L_2,L_3], \\ (L_2-L_3-L_1)/(2(L_2-L_1))(x-L_1)+L_1/2 & x\in[L_1,L_2]. \end{cases} \tag{4.37}$$

Lemma 5. *Let u be the solution of (1.1)$_1$. Then, the functional*

$$J_1(t) = -\int_\Omega q(x)u_x(x,t)u_t(x,t)\,dx,$$

satisfies the estimate

$$\begin{aligned}\frac{d}{dt}J_1(t) \leq \tfrac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \left(\tfrac{a}{2}+\delta_1\right)\int_\Omega u_x^2(x,t)\,dx + C(\delta_1)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx \\ -\tfrac{a}{4}[(L_3-L_2)u_x^2(L_2,t)+L_1u_x^2(L_1,t)].\end{aligned} \tag{4.38}$$

Proof. Taking the derivative of $J_1(x)$ with respect to t and using (1.1)$_1$, we obtain

$$\begin{aligned}&\frac{d}{dt}J_1(t) \\ &= -\int_\Omega q(x)u_{xt}(x,t)u_t(x,t)\,dx - a\int_\Omega q(x)u_x(x,t)u_{xx}(x,t)\,dx + \mu\int_\Omega q(x)u_x(x,t)\varpi(u_t(x,t))\,dx.\end{aligned}$$

Integrating by parts, we have

$$-\int_\Omega q(x)u_{xt}(x,t)u_t(x,t)\,dx = -\tfrac{1}{2}[q(x)u_t^2(x,t)]_{\partial\Omega} + \tfrac{1}{2}\int_\Omega q_x(x)u_t^2(x,t)\,dx. \tag{4.39}$$

On the other hand, then

$$-a\int_\Omega q(x)u_x(x,t)u_{xx}(x,t)\,dx = -\tfrac{a}{2}[q(x)u_x^2(x,t)]_{\partial\Omega} + \tfrac{a}{2}\int_\Omega q_x(x)u_x^2(x,t)\,dx. \tag{4.40}$$

By using the boundary conditions (1.2) we have

$$\tfrac{1}{2}[q(x)u_t^2(x,t)]_{\partial\Omega} = \tfrac{1}{4}L_1u_t^2(L_1,t) + \tfrac{1}{2}(L_3-L_2)u_t^2(L_2,t) \geq 0. \tag{4.41}$$

Also, we have

$$-\frac{a}{2}[q(x)u_x^2(x,t)]_{\partial\Omega} = -\frac{aL_1}{4}[u_x^2(L_1,t)-u_x^2(0,t)] - \frac{a(L_2-L_3)}{4}[u_x^2(L_3,t)-u_x^2(L_2,t)] \\ = -\frac{aL_1}{4}u_x^2(L_1,t) - \frac{a(L_3-L_2)}{4}u_x^2(L_2,t). \tag{4.42}$$

Using Young's inequality, we obtain

$$\mu\int_\Omega q(x)u_x(x,t)\varpi(u_t(x,t))\,dx \le \delta_1\int_\Omega u_x^2(x,t)\,dx + C(\delta_1)\mu^2\int_\Omega \varpi^2(u_t(x,t))\,dx. \tag{4.43}$$

Thus (4.38) follows from (4.39)–(4.43). This completes the proof. □

Lemma 6. *Let v be the solution of* $(1.1)_2$. *Then the functional*

$$J_2(t) = -\int_{L_1}^{L_2} q(x)v_x(x,t)v_t(x,t)\,dx,$$

satisfies, the estimate

$$\frac{d}{dt}J_2(t) \le q\frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2(x,t)\,dx + \int_{L_1}^{L_2} bv_x^2(x,t)\,dx\right) + \frac{b}{4}[(L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)]. \tag{4.44}$$

Proof. By the same method, taking the derivative of J_2 with respect to t and using $(1.1)_2$, we obtain

$$\frac{d}{dt}J_2(t) = -\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx - \int_{L_1}^{L_2} q(x)v_x(x,t)v_{tt}(x,t)\,dx \\ -\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx - b\int_{L_1}^{L_2} q(x)v_x(x,t)v_{xx}(x,t)\,dx. \tag{4.45}$$

Integrating by parts, we have

$$-\int_{L_1}^{L_2} q(x)v_{xt}(x,t)v_t(x,t)\,dx \\ = -\tfrac{1}{2}[q(x)v_t^2(x,t)]_{L_1}^{L_2} + \tfrac{1}{2}\int_{L_1}^{L_2} q_x(x)v_t^2(x,t)\,dx \\ = -\frac{L_2-L_3}{4}v_t^2(L_2,t) + \frac{L_1}{4}v_t^2(L_1,t) + \frac{L_2-L_3-L_1}{4(L_2-L_1)}\int_{L_1}^{L_2} v_t^2(x,t)\,dx, \tag{4.46}$$

and

$$-b\int_{L_1}^{L_2} q(x)v_x(x,t)v_{xx}(x,t)\,dx \\ = -\tfrac{b}{2}[q(x)v_x^2(x,t)]_{L_1}^{L_2} + \tfrac{b}{2}\int_{L_1}^{L_2} q_x(x)v_x^2(x,t)\,dx \\ = -b\tfrac{1}{4}(L_2-L_3)v_x^2(L_2,t) + b\tfrac{1}{4}L_1v_x^2(L_1,t) + b\frac{L_2-L_3-L_1}{4(L_2-L_1)}\int_{L_1}^{L_2} v_x^2(x,t)\,dx. \tag{4.47}$$

Estimate (4.44) follows by substituting (4.46) and (4.47) into (4.45). This completes the proof. □

We are now in position to define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy total functional E

Lemma 7. *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + \gamma J(t) + \gamma_1 J_1(t) + \gamma_2 J_2(t), \tag{4.48}$$

where N, γ, γ_1 and γ_2 are positive real numbers to be chosen appropriately later, satisfies

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \tag{4.49}$$

for two positive constants β_1 and β_2.

Proof. Let $L(t) = \gamma J(t) + \gamma_1 J_1(t) + \gamma_2 J_2(t)$

$$|L(t)| \le \gamma \int_\Omega |u(x,t)u_t(x,t)|\,dx + \gamma \int_{L_1}^{L_2} |v(x,t)v_t(x,t)|\,dx + \gamma_1 \int_\Omega |q(x)u_x(x,t)u_t(x,t)|\,dx$$
$$+\gamma_2 \int_{L_1}^{L_2} |q(x)v_x(x,t)v_t(x,t)|\,dx.$$

Exploiting Young and Poincaré's inequalities and (4.28), we obtain

$$\begin{aligned} |L(t)| \le{}& \tfrac{C_s}{2} \int_\Omega u_x^2(x,t)\,dx + \tfrac{1}{2} \int_\Omega u_t^2(x,t)\,dx + \tfrac{C_s}{2} \int_{L_1}^{L_2} v_x^2(x,t)\,dx + \tfrac{1}{2} \int_{L_1}^{L_2} v_t^2(x,t)\,dx \\ &+ \tfrac{c_1}{2} \int_\Omega u_x^2(x,t) + \tfrac{c_1}{2} \int_\Omega u_t^2(x,t)\,dx + \tfrac{c_2}{2} \int_{L_1}^{L_2} v_x^2(x,t) + \tfrac{c_2}{2} \int_{L_1}^{L_2} v_t^2(x,t)\,dx \\ \le{}& cE(t). \end{aligned} \tag{4.50}$$

Consequently, $|L(t) - NE(t)| \le cE(t)$, which yields

$$(N-c)E(t) \le \mathcal{L}(t) \le (N+c)E(t).$$

Choosing N large enough, we obtain estimate (4.49). This completes the proof. □

Lemma 8. *Let (u, v) be a solution of (1.1)–(1.3). Then $\mathcal{L}(t)$ satisfies the following estimate, along the solution and for some positive constants $m, c > 0$*

$$\frac{d}{dt}\mathcal{L}(t) \le -mE(t) + c \int_\Omega [u_t^2(x,t) + \varpi^2(u_t(x,t))]\,dx. \tag{4.51}$$

Proof. Taking the derivative of (4.48) with respect to t and making use of (4.32), (4.36), (4.38) and (4.44), we obtain

(4.52)
$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \le{}& (\gamma + \frac{\gamma_1}{2}) \int_\Omega u_t^2(x,t)dx \left\{\gamma(a - \delta C_s) - \gamma_1\left(\frac{a}{2} + \delta_1\right)\right\} \int_\Omega u_x^2(x,t)dx \\ &+ \left\{\gamma_2 \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)} + \gamma\right\} \int_{L_1}^{L_2} v_t^2(x,t)dx + b\left\{\gamma_2 \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)} + \gamma\right\} \int_{L_1}^{L_2} v_x^2(x,t)dx \\ &- \frac{a}{4}\left\{\gamma_1 - \gamma_2 \frac{b}{a}\right\}[(L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)] + \mu^2\{\gamma C(\delta) + \gamma_1 C(\delta_1)\} \int_\Omega \varpi^2(u_t(x,t))dx. \end{aligned}$$

At this point, we choose our constants in (4.52), carefully, so that all the coefficients in (4.52) will be negative. Indeed, under the assumption (4.29), we can always find γ, γ_1 and γ_2 such that

$$\gamma_2 \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)} + \gamma < 0, \quad \gamma_1 > \gamma_2 \frac{b}{a}, \quad \gamma > \frac{\gamma_1}{2},$$

we may δ and δ_1 small enough such that

$$\gamma \delta C_s + \gamma_1 \delta_1 < a\left(\gamma - \frac{\gamma_1}{2}\right).$$

Then

$$\frac{d}{dt}\mathcal{L}(t) \leq -mE(t) + c\int_\Omega [u_t^2(x,t) + \varpi^2(u_t(x,t))]\,dx.$$

This completes the proof. □

We are now ready to prove Theorem 4.1

Proof of Theorem 4.1. As in Komornik [7], we consider the following partition of Ω,

$$\Omega_1 = \{x \in \Omega : |u_t| > \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : |u_t| \leq \varepsilon\}.$$

Case 1. If G is linear on $[0, \varepsilon]$, then we deduce that

$$\mathcal{L}'(t) \leq -mE(t) + c\int_\Omega u_t(x,t)\varpi(u_t(x,t))\,dx \leq -mE(t) - cE'(t).$$

Consequently, we arrive at

$$(\mathcal{L}(t) + cE(t))' \leq -mE(t).$$

Recalling that

$$\mathcal{L}(t) + cE(t) \sim E(t),$$

we obtain

$$E(t) \leq c' e^{-c'' t}.$$

Case 2. If G is nonlinear on $[0, \varepsilon]$. In this case, we define

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t(x,t)\varpi(u_t(x,t))\,dx,$$

and exploit Jensen's inequality and the concavity of G^{-1} to obtain

$$G^{-1}(I(t)) \geq c\int_{\Omega_1} G^{-1}(u_t\varpi(u_t))\,dx, \tag{4.53}$$

by using (4.53) and (2.4), we obtain

$$\int_{\Omega_1} [u_t^2(x,t) + \varpi^2(u_t(x,t)]dx \leq \int_{\Omega_1} G^{-1}(u_t\varpi(u_t))\,dx \leq cG^{-1}(I(t)), \tag{4.54}$$

using (4.51) and (4.54), we have

$$\mathcal{L}'(t) \leq -mE(t) + cG^{-1}(I(t)). \tag{4.55}$$

We define F_0 by

$$F_0(t) = H'\left(\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + c_0 E(t).$$

Then, we see easily that, for $a_1, a_2 > 0$

$$a_1 F_0(t) \leq E(t) \leq a_2 F_0(t). \tag{4.56}$$

By recalling that $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \varepsilon]$ and making use of (4.28) and (4.55), we obtain

$$\begin{aligned} F_0'(t) &= \frac{E'(t)}{E(0)} G''\left(\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + G'\left(\frac{E(t)}{E(0)}\right)\mathcal{L}'(t) + c_0 E'(t) \\ &\leq -mE(t)G'\left(\frac{E(t)}{E(0)}\right) + cG'\left(\frac{E(t)}{E(0)}\right)G^{-1}(I(t)) + c_0 E'(t). \end{aligned} \tag{4.57}$$

Let G^* be the convex conjugate of G in the sense of Young

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)],$$

and G satisfies the generalized Young's inequality

$$AB \leq G^*(A) + G(B),$$

with $A = G'(E(t)/E(0))$ and $B = G^{-1}(I(t))$

$$\begin{aligned} F_0'(t) &\leq -mE(t)G'\left(\frac{E(t)}{E(0)}\right) + cG^*\left(G'\left(\frac{E(t)}{E(0)}\right)\right) + cI(t) + c_0 E'(t) \\ &\leq -mE(t)G'\left(\frac{E(t)}{E(0)}\right) + c\frac{E(t)}{E(0)}G'\left(\frac{E(t)}{E(0)}\right) - cE'(t) + c_0 E'(t). \end{aligned} \tag{4.58}$$

Choosing $c_0 > c$, we obtain

$$F_0'(t) \leq -k\frac{E(t)}{E(0)}G'\left(\frac{E(t)}{E(0)}\right) = -kG_1\left(\frac{E(t)}{E(0)}\right), \tag{4.59}$$

where $G_2(t) = tG'(t)$. Since

$$G_2'(t) = G'(t) + tG''(t),$$

and G is convex on $(0, \varepsilon]$ we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $(0, 1]$. By setting $F(t) = a_1 F_0(t)/E(0)$ (a_1 is given in (4.56)), we easily see, by (4.56), that

$$F(t) \sim E(t). \tag{4.60}$$

Using (4.59), we arrive at

$$F'(t) \leq -k_1 G_2(F(t)).$$

By recalling (4.31), we deduce

$$G_2(t) = -\frac{1}{G_1'(t)},$$

and

$$F'(t) \le \frac{k}{G_1'(F(t))}.$$

which gives

$$[G_1(F(t))]' = F'(t)G_1'(t) \le k_1.$$

A simple integration leads to

$$G_1(F(t)) \le k_1 t + G_1(F(0)).$$

Consequently,

$$F(t) \le G_1^{-1}(k_1 t + k_2). \tag{4.61}$$

Using (4.60) and (4.61) we obtain (4.30). The proof is completed. □

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