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Article

Stability for Weakly Coupled Wave Equations with a General Internal Control of Diffusive Type

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Abstract: The present paper deals with well-posedness and asymptotic stability for weakly coupled wave equations with a more general internal control of diffusive type. Owing to the semigroup theory of linear operator, the well-posedness of system is proved. Furthermore, we show a general decay rate result. The method is based on the frequency domain approach combined with multiplier technique.

Keywords: semigroup theory; wave coupled system; general decay

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1. Introduction

When describing the propagation of nonlinear waves with an internal control of diffusive type, the theory of semigroup is often used. It is used in the case, which is quite important for applications, when the internal diffusive mechanism is described by integer derivatives. The large amount of currently available experimental data on the internal structure of nonlinear waves in applications requires the complication and modification of mathematical modeling methods. Here, the main attention is paid to the construction and analysis of stability for nonlinear mathematical models that reflect the influence of internal control of diffusive type.

To begin with, let Ω be a bounded open domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $x \in \Omega, t \in (0, +\infty)$ and $\varpi \in (-\infty, +\infty)$. We consider the following system of coupled wave equations with general internal control of diffusive type

$$\begin{cases} \partial_{tt}u - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi,t)d\varpi + \beta v = 0, \\ \\ \partial_{tt}v - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi,t)d\varpi + \beta u = 0, \\ u = v = 0 & \text{on } \partial\Omega \\ \phi_t(x,\varpi,t) + (\varpi^2+\eta)\phi(x,\varpi,t) - \partial_t u \varrho(\varpi) = 0, \\ \varphi_t(x,\varpi,t) + (\varpi^2+\eta)\varphi(x,\varpi,t) - \partial_t v \varrho(\varpi) = 0, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), \\ v(x,0) = v_0(x), \partial_t v(x,0) = v_1(x), \\ \phi(x,\varpi,0) = \phi_0(x,\varpi) \text{ and } \varphi(x,\varpi,0) = \varphi_0(x,\varpi), \end{cases} \tag{1}$$

where $\zeta > 0, \eta \geq 0$ and ϱ are a general measure density, the initial data are taken in suitable spaces, and the coefficient β satisfies the condition

$$0 < |\beta| < \delta C$$

where $\delta \in (0,1)$. When $\varrho(\varpi) = |\varpi|^{\frac{2\alpha-1}{2}}, \zeta = \gamma\pi^{-1}\sin(\alpha\pi)$ and $\phi_0 \equiv 0$, problem $(1)_{1,2}$ becomes

$$\begin{cases} \partial_{tt}u - \Delta_x u + \gamma\partial_t^{\alpha,\eta}u + \beta v = 0, \\ \\ \partial_{tt}v - \Delta_x v + \gamma\partial_t^{\alpha,\eta}v + \beta u = 0, \end{cases}$$

where $\partial_t^{\alpha,\eta}$ denotes the generalized Caputo's fractional derivative of order α, $0 < \alpha < 1$ with respect to the time variable. It is defined by

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds,\ \eta \geq 0.$$

In [1], Mbodje studied the energy decay of the wave equation with a boundary control of fractional derivative type, that is, for $x \in (0,L), t \in (0,+\infty)$

$$\begin{cases} \partial_{tt}u(x,t) - u_{xx}(x,t) = 0, \\ u(0,t) = 0, \\ u_x(L,t) + \rho\partial_t^{\alpha,\eta}u(L,t) = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x). \end{cases}$$

A new approach named "diffusive representation" is used to solve the problem. The first model is transformed into a related system which can be easily treated by the energy method. If $\eta = 0$, the strong asymptotic stability of solutions is proved and, when $\eta \neq 0$, an algebraic decay rate $\mathcal{E}(t) \leq C/t$ for $t > 0$ is shown. In [2], Villagram et al. study the stabilization for the following coupled wave equations with dynamic control of fractional derivative type, for $x \in (0,1), t \in (0,+\infty)$

$$\begin{cases} \partial_{tt}u - u_{xx} + \beta v = 0, \\ \partial_{tt}v - v_{xx} + \beta u = 0, \\ u(0,t) = v(0,t) = 0, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x) \text{ and } v(x,0) = v_0(x), \partial_t v(x,0) = v_1(x), \\ u_x(1,t) = -\partial_t^{\alpha,\eta}u(1,t) \text{ and } v_x(1,t) = -\partial_t^{\alpha,\eta}v(1,t). \end{cases}$$

The authors proved that the decay of energy is not exponential, but it is polynomial.

Recently, in [3], Boudaoud and Benaissa extended the result of Mbodje to a higher-space dimension and general internal control of diffusive type.

$$\begin{cases} \partial_{tt}u - \Delta_x u + \zeta\int_{-\infty}^{+\infty}\varrho(\varpi)\phi(x,\varpi,t)d\varpi = 0, \\ u(x,t) = 0 & \text{on } \partial\Omega, \\ \phi_t(x,\varpi,t) + (\varpi^2+\eta)\phi(x,\varpi,t) - \partial_t u\varrho(\varpi) = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \\ \phi(x,\varpi,0) = \phi_0(x,\varpi), \end{cases}$$

The authors proved a very general rate depending on the form of the function ϱ.

Our paper extends all the previous works, and its plan is as follows. In Section 2, we give preliminary results, and we establish the well-posedness of the system (1), owing to the Hille–Yosida Theorem. We show, in Section 3, the lack of exponential stability. In Section 4, an asymptotic stability of our model is studied, where the main results are Theorem 4 and Theorem 7. In Theorem 7, we established a general rate of decay which depends on that of the density function ϱ.

Remark 1. *For this topic, we can say that there are many related problems which still are open, such as in the unbounded domain, where one can consider the same model in $\mathbb{R}^n$ with weighted functions.*

2. Preliminary Results and Well-Posedness

We state hypotheses on the even non-negative measurable function ϱ as

$$\int_{-\infty}^{\infty} \frac{\varrho(\varpi)^2}{1+\varpi^2}\, d\varpi < \infty. \tag{2}$$

Now, we recall some definitions which are needed in Section 4 for the application.

Definition 1. *Let $a \geq 0$, and let $M : [a, +\infty) \to (0, +\infty)$ be a measurable function, then M has a positive increase if there exist $\alpha > 0, c \in (0, 1]$ and $s_0 \geq a$, such that*

$$\frac{M(\kappa s)}{M(s)} \geq c\kappa^{\alpha}, \quad \kappa \geq 1, \quad s \geq s_0.$$

The next Lemma will be useful (see [1]).

Lemma 1. *Let*

$$D = \{\kappa \in \mathbb{C} / \Re\kappa + \eta > 0\} \cup \{\Im\kappa \neq 0\},$$

if $\kappa \in D$, then

$$\int_{-\infty}^{+\infty} \frac{\varrho^2(\varpi)}{\kappa + \eta + \varpi^2}\, d\varpi = \frac{\pi}{\sin \alpha\pi}(\kappa + \eta)^{\alpha - 1},$$

and

$$\int_{-\infty}^{+\infty} \frac{\varrho^2(\varpi)}{(\kappa + \eta + \varpi^2)^2}\, d\varpi = (1 - \alpha)\frac{\pi}{\sin \alpha\pi}(\kappa + \eta)^{\alpha - 2}.$$

We are now ready to give the existence and uniqueness result for the problem (1) by using semigroup theory. The energy space is defined as

$$\mathcal{H} = [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times [L^2(\Omega \times (-\infty, +\infty))]^2,$$

equipped with the following inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} &= \int_{\Omega} \left(w\overline{\tilde{w}} + z\overline{\tilde{z}} + \nabla_x u \nabla_x \overline{\tilde{u}} + \nabla_x v \nabla_x \overline{\tilde{v}} + \beta u \overline{\tilde{v}} + \beta v \overline{\tilde{u}} \right) dx \\ &+ \zeta \int_{\Omega} \int_{-\infty}^{+\infty} \left(\phi\overline{\tilde{\phi}} + \varphi\overline{\tilde{\varphi}} \right) d\varpi dx, \end{aligned} \tag{3}$$

where

$$U = (u, v, w, z, \phi, \varphi)^T, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z}, \tilde{\phi}, \tilde{\varphi})^T.$$

Remark 2. *Note that if $0 < |\beta| < \delta C$, we have*

$$\begin{aligned} 2\left|\beta\Re\langle u, \overline{v}\rangle\right| &\leq 2|\beta| \|u\|_2 . \|v\|_2 \\ &\leq 2|\beta| . \tfrac{1}{C} \|\nabla_x u\|_2 . \|\nabla_x v\|_2 \\ &\leq \delta(\|\nabla_x u\|_2 + \|\nabla_x v\|_2), \end{aligned} \tag{4}$$

which guarantees the positivity of the norm.

In order to transform the problem (1) to an abstract problem on the Hilbert space $\mathcal{H}$, we introduce the vector function $U = (u, v, w, z, \phi, \varphi)^T$, where $w = \partial_t u$ and $z = \partial_t v$. Then, problem (1) can be rewritten as

$$\partial_t U = \mathcal{A}U, \;\; U(0) = U_0, \tag{5}$$

where $U_0 = (u_0, v_0, u_1, v_1, \phi_0, \varphi_0)^T$, and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as follows

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ z \\ \phi, \\ \varphi \end{pmatrix} = \begin{pmatrix} w \\ z \\ \Delta_x u - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi - \beta v \\ \Delta_x v - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi - \beta u \\ -(\varpi^2+\eta)\phi + w(x)\varrho(\varpi) \\ -(\varpi^2+\eta)\varphi + z(x)\varrho(\varpi) \end{pmatrix}. \tag{6}$$

and its domain is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,w,z,\phi,\varphi)^T \text{ in } \mathcal{H} : u,v \in H^2(\Omega)\cap H_0^1(\Omega), w,z \in H_0^1(\Omega), \\ \Delta_x u(x) - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi - \beta v \in L^2(\Omega) \text{ and} \\ \Delta_x v(x) - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi - \beta u \in L^2(\Omega) \\ -(\varpi^2+\eta)\phi + w(x)\varrho(\varpi) \in L^2(\Omega\times(-\infty,+\infty)), |\varpi|\phi \in L^2(\Omega\times(-\infty,+\infty)) \\ -(\varpi^2+\eta)\varphi + z(x)\varrho(\varpi) \in L^2(\Omega\times(-\infty,+\infty)), |\varpi|\varphi \in L^2(\Omega\times(-\infty,+\infty)) \end{array} \right\}.$$

The energy associated to the solution of the problem (1) is given by

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\Big[\|\partial_t u\|_2^2 + \|\partial_t v\|_2^2 + \|\nabla_x u\|_2^2 + \|\nabla_x u\|_2^2\Big] \\ &+2\beta\int_\Omega uvdx + \frac{\zeta}{2}\int_\Omega\int_{-\infty}^{+\infty}\Big(|\phi(x,\varpi,t)|^2 + |\varphi(x,\varpi,t)|^2\Big)\,d\varpi dx. \end{aligned} \tag{7}$$

Differentiating $\mathcal{E}$ in a formal way, using (1) and integrating by parts, we obtain, after a straightforward computation, the following Lemma.

Lemma 2. *Let (u,v,w,z,ϕ,φ) be a regular solution of problem (1). Then, the energy functional defined by (7) satisfies*

$$\begin{aligned} \partial_t\mathcal{E}(t) &= -\zeta\int_\Omega\int_{-\infty}^{+\infty}(\varpi^2+\eta)\Big(|\phi(x,\varpi,t)|^2 + |\varphi(x,\varpi,t)|^2\Big)\,d\varpi dx \\ &\le 0. \end{aligned} \tag{8}$$

We have the following results.

Proposition 1. *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $\{\mathcal{S}(t)\}_{t\ge 0}$*

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative. We observe that $U \in D(\mathcal{A})$ and by (5), (8) and the fact that

$$\mathcal{E}(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{9}$$

we obtain

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta\int_\Omega\int_{-\infty}^{+\infty}(\varpi^2+\eta)|\phi(x,\varpi)|^2\,d\varpi dx. \tag{10}$$

In fact, using (3), and integrating by parts, we obtain

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= \int_\Omega (\nabla_x w \nabla_x \overline{u} - \overline{\nabla_x w \nabla_x \overline{u}} + \nabla_x z \nabla_x \overline{v} - \overline{\nabla_x z \nabla_x \overline{v}})dx \\
&- \zeta \int_{-\infty}^{+\infty} \left(\varpi^2 + \eta\right)\left(|\phi|^2 + |\varphi|^2\right) d\varpi \\
&+ \beta \int_\Omega (w\overline{v} - \overline{w\overline{v}} + z\overline{u} - \overline{z\overline{u}})dx + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\left[\left(\overline{\phi} z - \overline{\overline{\phi} z}\right) + \left(\overline{\phi} z - \overline{\overline{\phi} z}\right)\right] d\varpi \\
&= 2iIm \int_\Omega \nabla_x w \nabla_x \overline{u} dx + 2iIm \int_\Omega \nabla_x z \nabla_x \overline{v} dx \\
&+ 2i\beta Im \int_\Omega w \nabla_x \overline{v} dx + 2i\beta Im \int_\Omega z \nabla_x \overline{u} dx \\
&+ \zeta 2iIm \int_{-\infty}^{+\infty} \varrho(\varpi)\overline{\phi} z - \zeta \int_{-\infty}^{+\infty} \left(\varpi^2 + \eta\right)\left(|\phi|^2 + |\varphi|^2\right) d\varpi.
\end{aligned}$$

Hence, taking the real part, then estimate (10) holds.

Next, we prove that the operator $\kappa I - \mathcal{A}$ is surjective for every $\kappa > 0$. We show that for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, there exists a (unique) solution $U = (u, v, w, z, \phi, \varphi)^T \in D(\mathcal{A})$m such that $\kappa U - \mathcal{A}U = F$.

Then, in terms of components, the above equation reads

$$\begin{cases}
\kappa u - w = f_1, \\
\kappa v - z = f_2, \\
\kappa w - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi) d\varpi + \beta v = f_3, \\
\kappa z - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi) d\varpi + \beta u = f_4, \\
\kappa\phi + (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = f_5 \\
\kappa\varphi + (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = f_6.
\end{cases} \tag{11}$$

Suppose (u, v) is found with the appropriate regularity. Then, from $(11)_1$ and $(11)_2$, we find that

$$\begin{cases}
w = \kappa u - f_1 \in H_0^1(\Omega) \\
z = \kappa v - f_2 \in H_0^1(\Omega),
\end{cases} \tag{12}$$

and by $(11)_{5,6}$, we obtain

$$\begin{cases}
\phi = \frac{f_5(x,\varpi)}{\varpi^2+\eta+\kappa} + \frac{\kappa\varrho(\varpi)u(x)}{\varpi^2+\eta+\kappa} - \frac{\varrho(\varpi)f_1(x)}{\varpi^2+\eta+\kappa} \\
\varphi = \frac{f_6(x,\varpi)}{\varpi^2+\eta+\kappa} + \frac{\kappa\varrho(\varpi)v(x)}{\varpi^2+\eta+\kappa} - \frac{\varrho(\varpi)f_2(x)}{\varpi^2+\eta+\kappa}.
\end{cases} \tag{13}$$

On the other hand, replacing $(12)_{1,2}$ into $(11)_{3,4}$, respectively, yields

$$\begin{cases}
\kappa^2 u - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi) d\varpi + \beta v = f_3 + \kappa f_1 \\
\kappa^2 v - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi) d\varpi + \beta u = f_4 + \kappa f_2.
\end{cases} \tag{14}$$

Solving system (14) is equivalent to finding $u, v \in H^2(\Omega) \cap H_0^1(\Omega)$, such that

$$\begin{aligned}
&\int_\Omega (\kappa^2 u\overline{\tilde{u}} + \nabla_x u \nabla_x \overline{\tilde{u}})dx + \kappa\tilde{\zeta} \int_\Omega u\overline{\tilde{u}} dx + \beta \int_\Omega v\overline{\tilde{u}} \\
&= \int_\Omega (f_2 + \kappa f_1)\overline{\tilde{u}} dx - \zeta \int_{-\infty}^{+\infty} \frac{\varrho(\varpi)}{\varpi^2+\eta+\kappa} \int_\Omega f_5(x,\varpi)\overline{\tilde{u}} dx d\varpi + \tilde{\zeta} \int_\Omega f_1 \overline{\tilde{u}} dx,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\int_\Omega(\kappa^2 v\overline{\tilde{v}}+\nabla_x v\nabla_x\overline{\tilde{v}})dx+\kappa\tilde{\zeta}\int_\Omega v\overline{\tilde{v}}dx+\beta\int_\Omega u\overline{\tilde{v}}dx\\&=\int_\Omega(f_4+\kappa f_2)\overline{\tilde{v}}dx-\zeta\int_{-\infty}^{+\infty}\frac{\varrho(\varpi)}{\varpi^2+\eta+\kappa}\int_\Omega f_6(x,\varpi)\overline{\tilde{v}}dxd\varpi+\tilde{\zeta}\int_\Omega f_2\overline{\tilde{v}}dx,\end{aligned}\tag{16}$$

for all $\overline{\tilde{u}},\overline{\tilde{v}}\in H_0^1(\Omega)$ and $\tilde{\zeta}=\zeta\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta+\kappa}d\varpi$.

The system (15) and (16) is equivalent to the problem

$$\mathcal{B}((u,v),(\tilde{u},\tilde{v}))=\mathcal{L}(\tilde{u},\tilde{v}),\tag{17}$$

where the sesquilinear form

$$\mathcal{B}:[H_0^1(\Omega)\times H_0^1(\Omega)]^2\longrightarrow\mathbb{C},$$

and the antilinear form

$$\mathcal{L}:[H_0^1(\Omega)]^2\longrightarrow\mathbb{C},$$

are defined by

$$\mathcal{B}((u,v),(\tilde{u},\tilde{v}))=\int_\Omega(\kappa^2 u\overline{\tilde{u}}+\kappa^2 v\overline{\tilde{v}}+\nabla_x u\nabla_x\overline{\tilde{u}}+\nabla_x v\nabla_x\overline{\tilde{v}})dx+\kappa\tilde{\zeta}\int_\Omega(u\overline{\tilde{u}}+v\overline{\tilde{v}})dx,$$

and

$$\begin{aligned}\mathcal{L}(\tilde{u},\tilde{v})\;&=\;\int_\Omega(f_2+\kappa f_1)\overline{\tilde{u}}dx+\int_\Omega(f_4+\kappa f_2)\overline{\tilde{v}}dx+\tilde{\zeta}\int_\Omega f_1\overline{\tilde{u}}dx\\&-\;\zeta\int_{-\infty}^{+\infty}\frac{\varrho(\varpi)}{\varpi^2+\eta+\kappa}\int_\Omega(f_5(x,\varpi)\overline{\tilde{u}}+f_6(x,\varpi)\overline{\tilde{v}})dxd\varpi.\end{aligned}$$

It is not hard to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. By Lax–Milgram's Theorem, we deduce for all $\tilde{u},\tilde{v}\in H_0^1(\Omega)$, the problem (17) admits a unique solution $u,v\in H_0^1(\Omega)$. Using classical elliptic regularity, it follows from (15) and (16) that $u,v\in H^2(\Omega)\cap H_0^1(\Omega)$. In order to complete the existence of $U\in D(\mathcal{A})$, we need to prove $\phi,\varphi,|\varpi|\phi$ and $|\varpi|\varphi\in L^2(\Omega\times(-\infty,\infty))$. From $(13)_1$, we get

$$\int_\Omega\int_\mathbb{R}|\phi(\varpi)|^2\,d\varpi\,dx\le 3\int_\Omega\int_\mathbb{R}\frac{|f_5(x,\varpi)|^2}{(\varpi^2+\eta+\kappa)^2}\,d\varpi\,dx+3(\kappa^2\|u\|_2^2+\|f_1\|_2^2)\int_\mathbb{R}\frac{\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)^2}\,d\varpi.$$

Using (2), it easy to see that

$$\int_\mathbb{R}\frac{\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)^2}\,d\varpi\le\frac{1}{\kappa}\int_\mathbb{R}\frac{\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)}\,d\varpi<+\infty.$$

On the other hand, using the fact that $f_5\in L^2(\Omega\times(-\infty,\infty))$, we obtain

$$\int_\Omega\int_\mathbb{R}\frac{|f_5(x,\varpi)|^2}{(\varpi^2+\eta+\kappa)^2}\,d\varpi\,dx\le\frac{1}{\kappa^2}\int_\mathbb{R}|f_5(x,\varpi)|^2\,d\varpi\,dx<+\infty.$$

It follows that $\phi\in L^2(\Omega\times(-\infty,\infty))$. Next, using $(13)_1$, we obtain

$$\int_\Omega\int_\mathbb{R}|\varpi\phi(\varpi)|^2\,d\varpi\,dx\le 3\int_\Omega\int_\mathbb{R}\frac{|\varpi|^2|f_3(x,\varpi)|^2}{(\varpi^2+\eta+\kappa)^2}\,d\varpi\,dx+3(\kappa^2\|u\|^2+\|f_1\|^2)\int_\mathbb{R}\frac{|\varpi|^2\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)^2}\,d\varpi.$$

Using (2) again, it easy to see that

$$\int_\mathbb{R}\frac{|\varpi|^2\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)^2}\,d\varpi\le\int_\mathbb{R}\frac{\varrho^2(\varpi)}{(\varpi^2+\eta+\kappa)}\,d\varpi<+\infty.$$

Now, using the fact that $f_5 \in L^2(\Omega \times (-\infty, \infty))$, we find

$$\int_\Omega \int_{\mathbb{R}} \frac{|\varpi|^2 |f_5(\varpi)|^2}{(\varpi^2 + \eta + \kappa)^2} \, d\varpi \, dx \le \frac{1}{\kappa} \int_\Omega \int_{\mathbb{R}} |f_5(x, \varpi)|^2 \, d\varpi \, dx < +\infty.$$

It follows that $|\varpi|\phi \in L^2(\Omega \times (-\infty, \infty))$ and $\phi \in L^2(\Omega \times (-\infty, \infty))$. Finally, it is clear that

$$-(\varpi^2 + \eta)\phi(x, \varpi) + w(x)\varrho(\varpi) = \kappa\phi(x, \varpi) - f_5(x, \varpi) \in L^2(\Omega \times (-\infty, \infty)).$$

Using the same arguments, we can prove $\varphi, |\varpi|\varphi \in L^2(\Omega \times (-\infty, \infty))$. Then, $U \in D(\mathcal{A})$. Therefore, the operator $\kappa I - \mathcal{A}$ is surjective for any $\kappa > 0$. □

Consequently, using the Lumer–Philips Theorem [4], we have the following result.

Theorem 1 (Existence and uniqueness)**.** *If $U_0 \in \mathcal{H}$, then system (5) has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then system (5) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

3. Lack of Exponential Stability

Theorem 2 ([5])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space X. Then, $S(t)$ is exponentially stable if, and only if,*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{18}$$

and

$$\overline{\lim_{|\beta| \to \infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{19}$$

Our main result in this part is the following Theorem.

Theorem 3. *The semigroup generated by the operator $\mathcal{A}$ cannot be exponentially stable.*

Proof. Let $-\delta_n^2 = (i\delta_n)^2$ be a sequence of eigenvalues corresponding to the sequence of normalized eigenfunctions u_n of the operator Δ_x, such that $|\delta_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$$\begin{cases} \Delta_x u_n = -\delta_n^2 u_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Our aim is to prove, under some conditions, that if $i\delta_n$ satisfies (18), then (19) does not hold. In other words, we want to prove that there exist an infinite number of eigenvalues of $\mathcal{A}$ approaching the imaginary axis, which prevents the wave system (1) from being exponentially stable. Indeed, we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let κ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, w, z, \phi, \varphi)^T$. Then, $\mathcal{A}U = \kappa U$ is equivalent to

$$\begin{cases} \kappa u - w = 0, \\ \kappa v - z = 0, \\ \kappa w - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi) d\varpi + \beta v = 0, \\ \kappa z - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi) d\varpi + \beta u = 0, \\ \kappa\phi + (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = 0 \\ \kappa\varphi + (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = 0. \end{cases} \tag{21}$$

We note that assuming the decomposition given by $\Phi := u + v, \Theta := w + z$ and $\Lambda := \phi + \varphi$, we have

$$\begin{cases} \kappa\Phi - \Theta = 0, \\ \kappa\Theta - \Delta_x\Phi + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\Lambda d\varpi + \beta\Phi = 0, \\ \kappa\Lambda + (\varpi^2 + \eta)\Lambda - \Theta\varrho(\varpi) = 0. \end{cases} \tag{22}$$

The problem (22) can be rewritten as

$$V_t = \mathcal{A}_1 V, \;\; V(0) = V_0, \tag{23}$$

where $V_0 = (\Phi_0, \Phi_1, \Lambda_0)^T$, and $\mathcal{A}_1 : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ is defined as follows

$$\mathcal{A}_1(\Phi, \Theta, \Lambda) = \left(\Theta, \Delta_x\Phi - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\Lambda(\varpi)d\varpi, -(\varpi^2 + \eta)\Lambda + \Theta(x)\varrho(\varpi)\right). \tag{24}$$

Taking $\Psi := u - v, \Upsilon := w - z$ and $\Xi := \phi - \varphi$, we have

$$\begin{cases} \kappa\Psi - \Upsilon = 0, \\ \kappa\Upsilon - \Delta_x\Psi + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\Xi d\varpi - \beta\Psi = 0, \\ \kappa\Xi + (\varpi^2 + \eta)\Xi - \Upsilon\varrho(\varpi) = 0. \end{cases} \tag{25}$$

Moreover, note that

$$u := \frac{1}{2}(\Phi + \Psi), \; v := \frac{1}{2}(\Phi - \Psi), \; w := \frac{1}{2}(\Theta + \Upsilon), \; z := \frac{1}{2}(\Theta - \Upsilon), \varphi := \frac{1}{2}(\Lambda + \Xi),$$

and $\phi := \frac{1}{2}(\Lambda - \Xi)$. We define the Hilbert space

$$\mathbf{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (-\infty, +\infty)),$$

equipped with the following inner product

$$\begin{cases} \langle V_1, V_2 \rangle_{\mathbf{H}} &= \int_\Omega \left(\Theta_1\overline{\Theta_2} + \nabla_x\Phi_1\nabla_x\overline{\Phi_2} + \beta\Phi_1\overline{\Phi_2}\right)dx + \zeta \int_\Omega \int_{-\infty}^{+\infty} \Lambda_1\overline{\Lambda_2}d\varpi dx \\ \langle W_1, W_2 \rangle_{\mathbf{H}} &= \int_\Omega \left(\Upsilon_1\overline{\Upsilon_2} + \nabla_x\Psi_1\nabla_x\overline{\Psi_2} - \beta\Psi_1\overline{\Psi_2}\right)dx + \zeta \int_\Omega \int_{-\infty}^{+\infty} \Xi_1\overline{\Xi_2}d\varpi dx, \end{cases} \tag{26}$$

where $V_1 = (\Phi_1, \Theta_1, \Lambda_1), \; V_2 = (\Phi_2, \Theta_2, \Lambda_2), \; W_1 = (\Psi_1, \Upsilon_1, \Xi_1)$, and $W_2 = (\Psi_2, \Upsilon_2, \Xi_2)$ Note that inner product $\langle U_1, U_2 \rangle_{\mathcal{H}}$ given in (3) satisfies equality

$$\langle U_1, U_2 \rangle_{\mathcal{H}} = \frac{1}{2}\Big(\langle V_1, V_2 \rangle_{\mathbf{H}} + \langle W_1, W_2 \rangle_{\mathbf{H}}\Big).$$

Now, we need to solve problems (22)–(25). From $(22)_1$, we have

$$\Theta = \kappa\Phi. \tag{27}$$

Inserting (27) in $(22)_2$, we obtain

$$\kappa^2\Phi - \Delta_x\Phi + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\Lambda(\varpi)d\varpi + \beta\Phi = 0. \tag{28}$$

Then, from (27), $(22)_3$, and (28), we obtain

$$\kappa^2\Phi - \Delta_x\Phi + \zeta \int_{-\infty}^{+\infty} \kappa\Phi(x)\frac{\varrho^2(\varpi)}{\varpi^2+\eta+\kappa}d\varpi + \beta\Phi = 0, \tag{29}$$

it follows that

$$\Delta_x\Phi = \left(\kappa^2 + \beta + \zeta \int_{-\infty}^{+\infty} \kappa\frac{\varrho^2(\varpi)}{\varpi^2+\eta+\kappa}d\varpi\right)\Phi. \tag{30}$$

From (20) and (30), we obtain the existence of a sequence of eigenvalues κ_n of $\mathcal{A}$ corresponding to the sequence δ_n, such that

$$-\delta_n^2\Phi_n = \Delta_x\Phi_n = \left(\kappa_n^2 + \beta + \zeta\int_{-\infty}^{+\infty}\kappa_n\frac{\varrho^2(\varpi)}{\varpi^2+\eta+\kappa_n}d\varpi\right)\Phi_n,$$

then, we obtain

$$\delta_n^2 = -\kappa_n^2 - \beta - \zeta\int_{-\infty}^{+\infty}\kappa_n\frac{\varrho^2(\varpi)}{\varpi^2+\eta+\kappa_n}d\varpi.$$

By taking $\Lambda_n = \frac{\varrho(\varpi)}{\varpi^2+\eta+i\delta_n}\Phi_n$ and the vector $V_n = (\frac{\Phi_n}{i\delta_n}, \Phi_n, \Lambda_n)^T$, we have $V_n \in D(\mathcal{A}_1)$. Then, a direct computation gives

$$\mathcal{A}_1\begin{pmatrix}\frac{\Phi_n}{i\delta_n}\\ \Phi_n\\ \Lambda_n\end{pmatrix} = \begin{pmatrix}\Phi_n\\ i\delta_n\Phi_n + \beta\Phi_n - \zeta\int_{-\infty}^{+\infty}\varrho(\varpi)\Lambda_n(\varpi)d\varpi\\ i\delta_n\Lambda_n\end{pmatrix}.$$

It follows that

$$(i\delta_n I - \mathcal{A}_1)V_n = \begin{pmatrix}0\\ -\beta\Phi_n + \zeta\Phi_n\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta+i\delta_n}d\varpi\\ 0\end{pmatrix}.$$

Proving

$$\left\|(i\delta_n I - \mathcal{A}_1)^{-1}\right\|_{\mathcal{L}(\mathbf{H})} \longrightarrow \infty \quad \text{as} \quad |\delta_n| \longrightarrow \infty \quad \left(\text{i.e., as} \quad n \longrightarrow \infty\right),$$

reduces to show that, as $n \longrightarrow \infty$,

$$\left\| -\beta\Phi_n + \zeta\Phi_n\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta+i\delta_n}d\varpi\right\|_{L^2(\Omega)} \leq \left\|\zeta\Phi_n\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta+i\delta_n}d\varpi\right\|_{L^2(\Omega)} \longrightarrow 0.$$

Indeed, using the fact that

$$\left|\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta+i\beta_n}d\varpi\right| \leq\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

(see Lemma 4.3 in [6]) and the fact that Φ_n is a normalized eigenfunction of the operator Δ_x for each $n \in \mathbb{N}$, we obtain the desired limit. Therefore, taking $U = (u, v, w, z, \phi, \varphi) \in D(\mathcal{A})$, we conclude that

$$\|U\|_{\mathcal{H}}^2 = \frac{1}{2}\left(\|V\|_{\mathbf{H}}^2 + \|W\|_{\mathbf{H}}^2\right) \geq \frac{1}{2}\|V\|_{\mathbf{H}}^2 \longrightarrow +\infty.$$

This completes the proof. □

4. Stability

4.1. Strong Stability of the System

Here, we use the general Theorem of Arendt–Batty in [7] to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (1). Our main result is stated in the following

Theorem 4. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (5) satisfies*

$$\lim_{t\to\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

In order to prove Theorem 4, we need the following two Lemmas.

Lemma 3. *$\mathcal{A}$ does not have eigenvalues in $i\mathbb{R}$.*

Proof. **Step 1:** By contraction, we suppose that there exists $\kappa \in \mathbb{R}$, $\kappa \neq 0$ and $U \neq 0$, such that

$$\mathcal{A}U = i\kappa U. \tag{31}$$

Then, we obtain

$$\begin{cases} i\kappa u - w = 0, \\ i\kappa v - z = 0, \\ i\kappa w - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi + \beta v = 0, \\ i\kappa z - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi + \beta u = 0, \\ i\kappa\phi + (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = 0 \\ i\kappa\phi + (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = 0. \end{cases} \tag{32}$$

Now, using (31) and (10), we deduce that

$$\phi = 0 \quad \text{and} \quad \varphi = 0 \quad \text{in} \quad \Omega \times (-\infty, +\infty). \tag{33}$$

From $(32)_5$ and $(32)_1$, we have

$$w = 0 \quad \text{and} \quad u = 0 \quad \text{in} \quad \Omega. \tag{34}$$

It follows from $(32)_6$ and $(32)_2$ that we obtain

$$v = 0 \quad \text{and} \quad z = 0 \quad \text{in} \quad \Omega. \tag{35}$$

Therefore, $U = (u, v, w, z, \phi, \varphi)^T = 0$.
Step 2: $\kappa = 0$. The system (32) becomes

$$\begin{cases} w = 0, \\ z = 0, \\ \Delta_x u - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi - \beta v = 0, \\ \Delta_x v - \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi - \beta u = 0, \\ (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = 0 \\ (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = 0. \end{cases} \tag{36}$$

Hence, From $(36)_{1,2}$ and $(36)_{5,6}$, we obtain

$$w = 0, \ z = 0, \ \phi = 0 \quad \text{and} \quad \varphi = 0 \quad \text{in} \quad \Omega. \tag{37}$$

Multiplying $(36)_3$ by $\overline{u}$, $(36)_4$ by $\overline{v}$, and using integration by parts over Ω , we obtain

$$\begin{cases} \int_\Omega |\nabla_x u|^2 dx - \beta \int_\Omega v\overline{u} dx = 0, \\ \int_\Omega |\nabla_x v|^2 dx - \beta \int_\Omega u\overline{v} dx = 0. \end{cases} \tag{38}$$

Adding $(38)_1$ and $(38)_2$, and using (5.20), we have

$$\begin{aligned} \int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2) dx &\leq 2\beta \left| \int_\Omega v.u dx \right|, \\ &\leq \delta \int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2) dx. \end{aligned} \tag{39}$$

Consequently,

$$(1 - \delta) \int_\Omega \Big(|\nabla_x u|^2 + |\nabla_x v|^2 \Big) dx \leq 0. \tag{40}$$

Hence, u, v are constant in the whole domain Ω, and $u = v = 0$ on $\partial\Omega$, then we have $u = 0$, and $v = 0$ in the whole domain Ω. Therefore, $U = (u, v, w, z, \phi, \varphi)^T = 0$. We deduce that, consequently, $\mathcal{A}$ has no eigenvalue on the imaginary axis. $\square$

Lemma 4. *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0, \quad i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0.$$

Proof. We should prove that the operator $i\kappa I - \mathcal{A}$ is surjective for $\kappa \neq 0$. To this end, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$; we seek the $U = (u, v, , w, z, \phi, \varphi)^T \in D(\mathcal{A})$ solution of $(i\kappa I - \mathcal{A})U = F$.

Equivalently, we have

$$\begin{cases} i\kappa u - w = f_1, \\ i\kappa v - z = f_2, \\ i\kappa w - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi + \beta v = f_3, \\ i\kappa z - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi + \beta u = f_4, \\ i\kappa\phi + (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = f_5 \\ i\kappa\varphi + (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = f_6. \end{cases} \tag{41}$$

Inserting $(41)_{1,2}$ in $(41)_{3,4}$, respectively, we have

$$\begin{cases} -\kappa^2 u - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi + \beta v = f_3 + i\kappa f_1, \\ -\kappa^2 v - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi + \beta u = f_4 + i\kappa f_2. \end{cases} \tag{42}$$

Solving system (42) is equivalent to finding $u, v \in H^2(\Omega) \cap H_0^1(\Omega)$, such that

$$\begin{aligned} &\int_\Omega (-\kappa^2 u\bar{\tilde{u}} + \nabla_x u \nabla_x \bar{\tilde{u}})dx + i\kappa\tilde{\zeta} \int_\Omega u\bar{\tilde{u}}dx + \beta \int_\Omega v\bar{\tilde{u}}dx \\ &= \int_\Omega (f_3 + i\kappa f_1)\bar{\tilde{u}}dx - \zeta \int_{-\infty}^{+\infty} \frac{\varrho(\varpi)}{\varpi^2 + \eta + i\kappa} \int_\Omega f_5(x, \varpi)\bar{\tilde{u}}dxd\varpi + \tilde{\zeta} \int_\Omega f_1\bar{\tilde{u}}dx. \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\int_\Omega (-\kappa^2 v\bar{\tilde{v}} + \nabla_x v \nabla_x \bar{\tilde{v}})dx + i\kappa\tilde{\zeta} \int_\Omega v\bar{\tilde{v}}dx + \beta \int_\Omega u\bar{\tilde{v}}dx \\ &= \int_\Omega (f_3 + i\kappa f_1)\bar{\tilde{u}}dx - \zeta \int_{-\infty}^{+\infty} \frac{\varrho(\varpi)}{\varpi^2 + \eta + i\kappa} \int_\Omega f_6(x, \varpi)\bar{\tilde{v}}dxd\varpi + \tilde{\zeta} \int_\Omega f_2\bar{\tilde{v}}dx. \end{aligned} \tag{44}$$

for all $\bar{\tilde{u}}, \bar{\tilde{v}} \in H_0^1(\Omega)$.

The system (43) and (44) is equivalent to the problem

$$-\langle L_\kappa(u, v), (\tilde{u}, \tilde{v})\rangle + a((u, v), (\tilde{u}, \tilde{v})) = \mathcal{L}(\tilde{u}, \tilde{v}), \tag{45}$$

where

$$a((u, v), (\tilde{u}, \tilde{v})) = \int_\Omega \left(\nabla_x u \nabla_x \bar{\tilde{u}} + \nabla_x v \nabla_x \bar{\tilde{v}}\right) dx + i\kappa\tilde{\zeta} \int_\Omega \left(u\bar{\tilde{u}} + v\bar{\tilde{v}}\right) dx + \beta \int_\Omega (u\bar{\tilde{v}} + v\bar{\tilde{u}})dx,$$

and

$$\langle L_\kappa(u, v), (\tilde{u}, \tilde{v})\rangle_{[H_0^1(\Omega)]^2} = \int_\Omega \kappa^2 \left(u\bar{\tilde{u}} + v\bar{\tilde{v}}\right) dx.$$

Owing to the compactness of embedding $L^2(\Omega)$ into $H^{-1}(\Omega)$, and from $H_0^1(\Omega)$ into $L^2(\Omega)$, it follows that the operator L_κ is compact from $L^2(\Omega)$ into $L^2(\Omega)$. This way, by the Fredholm alternative, proving the existence of a solution (u, v) of (45) reduces to show that 1 is not an eigenvalue of L_κ for $\mathcal{L} \equiv 0$. Indeed, if there exists $u \neq 0$ and $v \neq 0$, such that

$$\langle L_\kappa(u, v), (\tilde{u}, \tilde{v})\rangle_{[H_0^1(\Omega)]^2} = a_{[H_0^1(\Omega)]^2}((u, v), (\tilde{u}, \tilde{v})) \quad \forall \tilde{u}, \tilde{v} \in H_0^1(\Omega),$$

then $i\kappa$ is an eigenvalue of $\mathcal{A}$. Therefore, from Lemma 3, we deduce that $u = 0$.

Now, if $\kappa = 0$ and $\eta \neq 0$, by using the Lax–Milgram Lemma, we obtain the result. □

Proof. (Of Theorem 4.) Following a general Theorem of Arendt–Batty in [7], the C_0-semigroup of contractions can be taken as strongly stable if $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$ and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is at most a countable set. Owing to the Lemmas 3 and 4, we find the result. □

4.2. General Decay

Theorem 5 ([8]). *Let $\mathcal{A}$ be the generator of a bounded C_0-semigroup $(S(t))_{t\geq 0}$ on X. Let X be a Banach space, if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq M(|\beta|),$$

where

$$M : \mathbb{R}_+ \to (0, \infty)$$

is a continuous nondecreasing function, then

$$\|e^{\mathcal{A}t}U_0\| \leq \frac{C}{M_{\log}^{-1}(ct)}\|U_0\|_{D(\mathcal{A})}, C, c > 0,$$

where

$$M_{\log} : \mathbb{R}_+ \to (0, \infty),$$

is defined by

$$M_{\log}(s) = M(s)(\log(1 + M(s)) + \log(1 + s)), \ \ s \geq 0.$$

We have the next important Theorem.

Theorem 6 ([9]). *Let $\mathcal{A}$ be the generator of a bounded C_0-semigroup $(S(t))_{t\geq 0}$ on X. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq M(|\beta|),$$

where X is a Hilbert space and

$$M : \mathbb{R}_+ \to (0, \infty)$$

is a continuous nondecreasing function of positive increase, then

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{M^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t \to \infty,$$

for a positive constant $C > 0$.

Theorem 7. *Let*

$$\mathcal{M}(\kappa) = c\mathcal{S}^{-2}\Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{(|\kappa| + \varpi^2 + \eta)^2}\Big)$$

for a suitable positive constant c, and where $\mathcal{S} = \int_{-\infty}^{+\infty} \frac{|\varpi|\varrho(\varpi)}{(|\kappa|+\varpi^2+\eta)^2}\, d\varpi$. Then, $S_{\mathcal{A}}(t)_{t\geq 0}$ satisfy

(1) *If $\mathcal{M}$ is a nondecreasing function of positive increase, then*

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{\mathcal{M}^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t \to \infty,$$

where $\mathcal{M}^{-1}$ is any asymptotic inverse of $\mathcal{M}$.

(2) *Let l be a nondecreasing slowly varying function, if*

$$\mathcal{M}(\kappa) \sim cl(|\kappa|), |\kappa| \to \infty,$$

then

$$\|e^{\mathcal{A}t}U_0\| \leq \frac{1}{l_{\log}^{-1}(ct)}\|U_0\|_{D(\mathcal{A})},$$

where

$$l_{\log}(s) = l(s)(\log(1 + l(s)) + \log(1 + s)), \ \ 0 \geq s.$$

Proof. We need to study the resolvent equation

$$(i\kappa I - \mathcal{A})U = F,$$

for $\kappa \in \mathbb{R}$, namely

$$\begin{cases} i\kappa u - w = f_1, \\ i\kappa v - z = f_2, \\ i\kappa w - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(\varpi)d\varpi + \beta v = f_3, \\ i\kappa z - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(\varpi)d\varpi + \beta u = f_4, \\ i\kappa\phi + (\varpi^2 + \eta)\phi - w(x)\varrho(\varpi) = f_5 \\ i\kappa\varphi + (\varpi^2 + \eta)\varphi - z(x)\varrho(\varpi) = f_6. \end{cases} \tag{46}$$

where

$$F = (f_1, f_2, f_3, f_4, f_5, f_6)^T.$$

Taking the inner product in $\mathcal{H}$ with

$$U = (u, v, w, z, \phi, \varphi)^T,$$

and using (10), we obtain

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{47}$$

This implies that

$$\zeta \int_\Omega \int_{-\infty}^{+\infty} (\varpi^2 + \eta)\big(|\phi(x,\varpi)|^2 + |\varphi(x,\varpi)|^2\big)\, d\varpi\, dx \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{48}$$

From $(46)_5$, we obtain

$$w(x)\varrho(\varpi) = (i\kappa + \varpi^2 + \eta)\phi(x,\varpi) - f_5(x,\varpi), \quad \forall (x,\varpi) \in \Omega \times (-\infty, +\infty). \tag{49}$$

By multiplying (49) by $(i\kappa + \varpi^2 + \eta)^{-2}|\varpi|$, we obtain

$$(i\kappa + \varpi^2 + \eta)^{-2} w(x)|\varpi|\varrho(\varpi) = (i\kappa + \varpi^2 + \eta)^{-1}|\varpi|\phi - (i\kappa + \varpi^2 + \eta)^{-2}|\varpi| f_5(x,\varpi), \quad x \in \Omega. \tag{50}$$

Hence, by taking the absolute values of both sides of (50) and applying triangle inequality, we obtain

$$|(i\kappa + \varpi^2 + \eta)^{-2}|\, |w(x)|\, |\varpi|\, \varrho(\varpi) \leq |(i\kappa + \varpi^2 + \eta)^{-1}|\, |\varpi|\, |\phi| + |(i\kappa + \varpi^2 + \eta)^{-2}|\, |\varpi|\, |f_5(x,\varpi)|.$$

By integration over $(-\infty, +\infty)$, we obtain

$$|w(x)| \left| \int_{-\infty}^{+\infty} \frac{|\varpi|\varrho(\varpi)}{(i\kappa + \varpi^2 + \eta)^2}\, d\varpi \right| \leq \left| \int_{-\infty}^{+\infty} \frac{|\varpi|\phi}{i\kappa + \varpi^2 + \eta}\, d\varpi \right| + \left| \int_{-\infty}^{+\infty} \frac{|\varpi| f_5(x,\varpi)}{(i\kappa + \varpi^2 + \eta)^2}\, d\varpi \right|. \tag{51}$$

On the other hand, by applying Cauchy–Schwartz inequality, we deduce that

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} \frac{|\varpi|\phi}{i\kappa + \varpi^2 + \eta}\, d\varpi \right| &\leq \left(\int_{-\infty}^{+\infty} |\varpi|^2 \phi^2\, d\varpi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left| \frac{1}{(i\kappa + \varpi^2 + \eta)^2} \right| d\varpi \right)^{\frac{1}{2}} \\ &\leq \left(\int_{-\infty}^{+\infty} (|\varpi|^2 + \eta)\phi^2\, d\varpi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \frac{d\varpi}{|i\kappa + \varpi^2 + \eta|^2} \right)^{\frac{1}{2}} \end{aligned} \tag{52}$$

and

$$\left| \int_{-\infty}^{+\infty} \frac{|\varpi| f_5(x,\varpi)}{(i\kappa + \varpi^2 + \eta)^2}\, d\varpi \right| \leq \left(\int_{-\infty}^{+\infty} |f_5(x,\varpi)|^2\, d\varpi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{|i\kappa + \varpi^2 + \eta|^4}\, d\varpi \right)^{\frac{1}{2}}. \tag{53}$$

By substituting (52) and (53) into (51), taking the square of inequality (51) and using the inequality $2AB \le A^2 + B^2$, we obtain

$$
\begin{aligned}
&|w(x)|^2 \Big| \int_{-\infty}^{+\infty} \frac{|\varpi|\varrho(\varpi)}{(i\kappa+\varpi^2+\eta)^2}\, d\varpi \Big|^2 \\
&\le 2\Big(\int_{-\infty}^{+\infty} (|\varpi|^2+\eta)|\phi|^2\, d\varpi \Big) \Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{|i\kappa+\varpi^2+\eta|^2} \Big) \\
&+2\Big(\int_{-\infty}^{+\infty} |f_5(x,\varpi)|^2\, d\varpi \Big) \Big(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{|i\kappa+\varpi^2+\eta|^4}\, d\varpi \Big).
\end{aligned} \tag{54}
$$

Integrating (54) over Ω, we obtain

$$
\begin{aligned}
&\Big(\int_\Omega |w(x)|^2\, dx \Big) \Big| \int_{-\infty}^{+\infty} \frac{|\varpi|\varrho(\varpi)}{(i\kappa+\varpi^2+\eta)^2}\, d\varpi \Big|^2 \\
&\le 2 \int_\Omega \int_{-\infty}^{+\infty} (|\varpi|^2+\eta)|\phi|^2\, d\varpi\, dx \Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{|i\kappa+\varpi^2+\eta|^2} \Big) \\
+ \quad & 2\Big(\int_\Omega \int_{-\infty}^{+\infty} |f_5(x,\varpi)|^2\, d\varpi\, dx \Big) \Big(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{|i\kappa+\varpi^2+\eta|^4}\, d\varpi \Big).
\end{aligned} \tag{55}
$$

Now, from Proposition 2.4 in [2],

$$
\begin{aligned}
& \Big| \int_{-\infty}^{+\infty} (i\kappa+\eta+\varpi^2)^{-2} |\varpi|\varrho(\varpi)\, d\varpi \Big| \\
\ge \quad & \frac{\sqrt{1+\cos\theta}}{\sqrt{2}} \int_{-\infty}^{+\infty} (|i\kappa+\eta|+\varpi^2)^{-2} |\varpi|\varrho(\varpi)\, d\varpi \\
\ge \quad & \frac{\sqrt{1+\cos\theta}}{\sqrt{2}} \int_{-\infty}^{+\infty} (|\kappa|+\varpi^2+\eta)^{-2} |\varpi|\varrho(\varpi)\, d\varpi,
\end{aligned}
$$

where $\cos\theta = \eta / \sqrt{\kappa^2+\eta^2}$. We obtain

$$
\Big| \int_{-\infty}^{+\infty} \tfrac{|\varpi|\varrho(\varpi)}{(i\kappa+\varpi^2+\eta)^2}\, d\varpi \Big| \ge \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \tfrac{|\varpi|\varrho(\varpi)}{(|\kappa|+\varpi^2+\eta)^2}\, d\varpi. \tag{56}
$$

Denoting $\mathcal{S} = \int_{-\infty}^{+\infty} \frac{|\varpi|\varrho(\varpi)}{(|\kappa|+\varpi^2+\eta)^2}\, d\varpi$, and by using (55), (56) and (48), we obtain

$$
\begin{aligned}
\mathcal{S}^2 \|w\|^2_{L^2(\Omega)} \quad &\le \quad 4\Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{|i\kappa+\varpi^2+\eta|^2} \Big) \|U\|\, \|F\| \\
&+ \quad 4\|f_5\|^2_{L^2(\Omega\times(-\infty,+\infty))} \Big(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{|i\kappa+\varpi^2+\eta|^4}\, d\varpi \Big) \\
&\le \quad 8\Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{(|\kappa|+\varpi^2+\eta)^2} \Big) \|U\|\, \|F\| \\
&+ \quad 16\|f_5\|^2_{L^2(\Omega\times(-\infty,+\infty))} \Big(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{(|\kappa|+\varpi^2+\eta)^4}\, d\varpi \Big).
\end{aligned} \tag{57}
$$

Using the same argument, we can prove

$$
\begin{aligned}
\mathcal{S}^2 \|z\|^2_{L^2(\Omega)} \quad &\le \quad 8\Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{(|\kappa|+\varpi^2+\eta)^2} \Big) \|U\|\, \|F\| \\
&+ \quad 16\|f_6\|^2_{L^2(\Omega\times(-\infty,+\infty))} \Big(\int_{-\infty}^{+\infty} \frac{|\varpi|^2}{(|\kappa|+\varpi^2+\eta)^4}\, d\varpi \Big).
\end{aligned} \tag{58}
$$

We now state the following

$$\mathcal{E}_u = \int_\Omega \Big(|w(x)|^2 + |z(x)|^2 + |\nabla_x u(x)|^2 + |\nabla_x v(x)|^2 \Big)\, dx.$$

Multiplying $(46)_3$ by $\bar{u}$ and $(46)_4$ by $\bar{v}$ leads to

$$\int_\Omega i\kappa w\bar{u}\, dx - \int_\Omega \Delta_x u\bar{u}\, dx + \int_\Omega \zeta\bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\, dx + \beta\int_\Omega v\bar{u}dx = \int_\Omega f_3\bar{u}\, dx,$$

and

$$\int_\Omega (i\kappa v\bar{v} - \Delta_x v\bar{v})dx + \int_\Omega \zeta\bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\, dx + \beta\int_\Omega u\bar{v}dx = \int_\Omega f_4\bar{v}dx.$$

Then,

$$\begin{cases} -\int_\Omega w\,(\overline{i\kappa u})\, dx + \int_\Omega |\nabla_x u|^2\, dx + \zeta\int_\Omega \bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\, dx + \beta\int_\Omega v\bar{u}dx = \int_\Omega f_2\bar{u}\, dx \\ -\int_\Omega z\,(\overline{i\kappa v})\, dx + \int_\Omega |\nabla_x u|^2\, dx + \zeta\int_\Omega \bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\, dx + \beta\int_\Omega u\bar{v}dx = \int_\Omega f_4\bar{v}dx. \end{cases} \quad (59)$$

Replacing $(46)_1$ into $(59)_1$ and $(46)_2$ into $(59)_2$, we have

$$\begin{cases} -\int_\Omega w\,(\bar{w} + \bar{f}_1)\, dx + \int_\Omega |\nabla_x u|^2\, dx + \zeta\int_\Omega \bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\, dx + \beta\int_\Omega v\bar{u}dx = \int_\Omega f_3\bar{u}\, dx \\ -\int_\Omega z\,(\bar{z} + \bar{f}_2)\, dx + \int_\Omega |\nabla_x v|^2\, dx + \zeta\int_\Omega \bar{u} \int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\, dx + \beta\int_\Omega u\bar{v}dx = \int_\Omega f_4\bar{v}dx. \end{cases}$$

Then,

$$\begin{aligned} &-\int_\Omega (|w(x)|^2 + |z(x)|^2)\, dx + \int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2)\, dx + \beta\int_\Omega (u\bar{v} + v\bar{u})\, dx \\ &+\zeta\int_\Omega \bar{u}\int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\, dx + \zeta\int_\Omega \bar{u}\int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\, dx = \int_\Omega (f_3\bar{u} + w\bar{f}_1 + f_4\bar{v} + \bar{f}_2 z)\, dx. \end{aligned}$$

It can be written as

$$\begin{aligned} &\int_\Omega (|w(x)|^2 + |z(x)|^2)\, dx + \int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2)\, dx \\ = \;& -\zeta\int_\Omega \bar{u}\int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\, dx - \zeta\int_\Omega \bar{v}\int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\, dx \\ + \;& \int_\Omega (f_3\bar{u} + w\bar{f}_1 + f_4\bar{v} + \bar{f}_2 z)\, dx - \beta\int_\Omega (u\bar{v} + v\bar{u})\, dx + 2\int_\Omega (|w(x)|^2 + |z(x)|^2)\, dx. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{E}_u \;\leq\;& \zeta\|u\|_2\Big(\int_\Omega \Big|\int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\Big|^2 dx\Big)^{\frac{1}{2}} + \zeta\|v\|_2\Big(\int_\Omega \Big|\int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\Big|^2 dx\Big)^{\frac{1}{2}} \\ &+\|f_3\|_2\,\|u\|_2 + \|w\|_2\,\|f_1\|_2 + \|f_4\|_2\,\|v\|_2 + \|f_2\|_2\,\|z\|_2 + 2\|w\|_2^2 + 2\|z\|_2^2 \\ &+\delta(\|\nabla_x u\|_2^2 + \|\nabla_x u\|_2^2), \end{aligned}$$

and using

$$\Big|\int_{-\infty}^{+\infty} \varrho(\varpi)\phi(x,\varpi)d\varpi\Big|^2 \leq \Big(\int_{-\infty}^{+\infty} \frac{\varrho^2(\varpi)}{\varpi^2+\eta}d\varpi\Big)\Big(\int_{-\infty}^{+\infty} (\varpi^2+\eta)|\phi|^2 d\varpi\Big),$$

and

$$\Big|\int_{-\infty}^{+\infty} \varrho(\varpi)\varphi(x,\varpi)d\varpi\Big|^2 \leq \Big(\int_{-\infty}^{+\infty} \frac{\varrho^2(\varpi)}{\varpi^2+\eta}d\varpi\Big)\Big(\int_{-\infty}^{+\infty} (\varpi^2+\eta)|\varphi|^2 d\varpi\Big),$$

we deduce that

$$\begin{aligned}\mathcal{E}_u &\leq \zeta\|u\|_2\Big(\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta}d\varpi\Big)^{\frac{1}{2}}\Big(\int_\Omega\int_{-\infty}^{+\infty}(\varpi^2+\eta)|\phi|^2 d\varpi\,dx\Big)^{\frac{1}{2}}\\ &+\zeta\|v\|_2\Big(\int_{-\infty}^{+\infty}\frac{\varrho^2(\varpi)}{\varpi^2+\eta}d\varpi\Big)^{\frac{1}{2}}\Big(\int_\Omega\int_{-\infty}^{+\infty}(\varpi^2+\eta)|\varphi|^2 d\varpi\,dx\Big)^{\frac{1}{2}}\\ &+\|f_3\|_2\,\|u\|_2+\|w\|_2\,\|f_1\|_2+\|f_4\|_2\,\|v\|_2+\|f_2\|_2\,\|z\|_2\\ &+2\|w\|_2^2+2\|z\|_2^2+\delta(\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{E}_u &\leq \varepsilon\|u\|_2^2+c(\varepsilon)\|U\|\|F\|+\varepsilon\|u\|_2^2+c(\varepsilon)\|f_3\|_2^2+\varepsilon\|v\|_2^2+c(\varepsilon)\|f_4\|_2^2\\ &+\|f_1\|_2^2+\|f_2\|_2^2+c\|w\|_2^2+c\|z\|_2^2+\delta(\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2).\end{aligned}$$

Using the estimation

$$c(\varepsilon)\|f_2\|_2^2+\|f_1\|_2^2+c(\varepsilon)\|f_4\|_2^2+\|f_2\|_2^2\leq c\|F\|^2,$$

and the classical Poincaré's inequality

$$\|u\|_2^2\leq c\|\nabla_x u\|_2^2 \quad \text{and} \quad \|v\|_2^2\leq c\|\nabla_x v\|_2^2,$$

we obtain

$$\mathcal{E}_u\leq 2\,\varepsilon\,c(\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2)+c(\|w\|_2^2+\|z\|_2^2)+c\|F\|^2+c\|U\|\,\|F\|.$$

Then, we obtain

$$\mathcal{E}_u\leq c\|w\|_2^2+\|z\|_2^2+c'\|F\|^2+c''\|U\|\,\|F\|,$$

and from (48), it follows that

$$\begin{aligned}\|\phi\|_{L^2}^2+\|\varphi\|_{L^2}^2 &= \int_\Omega\int_{-\infty}^{+\infty}\Big(|\phi|^2+|\varphi|^2\Big)d\varpi\,dx\\ &\leq C\int_\Omega\int_{-\infty}^{+\infty}(\varpi^2+\eta)\Big(|\phi|^2+|\varphi|^2\Big)d\varpi\,dx\leq C\|U\|\,\|F\|.\end{aligned}$$

We conclude that

$$\|U\|^2\leq c\|w\|_2^2+\|z\|_2^2+c'\|F\|^2+c''\|U\|\,\|F\|. \tag{60}$$

Inserting (57) into (60), we obtain

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 &\leq c\mathcal{S}^{-2}\Big(\int_{-\infty}^{+\infty}\frac{d\varpi}{(|\kappa|+\varpi^2+\eta)^2}\Big)\|U\|\,\|F\|\\ &+ c'\mathcal{S}^{-2}\|F\|^2\Big(\int_{-\infty}^{+\infty}\frac{|\varpi|^2}{(|\kappa|+\varpi^2+\eta)^4}\,d\varpi\Big)\\ &+ c''\|F\|^2+c'''\|U\|\,\|F\|.\end{aligned}$$

Then, we obtain

$$\|U\|_{\mathcal{H}}^2\leq\mathcal{M}^2(\kappa)\|F\|_{\mathcal{H}}^2, \tag{61}$$

where

$$\mathcal{M}(\kappa) = c\mathcal{S}^{-2}\Big(\int_{-\infty}^{+\infty} \frac{d\varpi}{(|\kappa| + \varpi^2 + \eta|)^2}\Big).$$

It follows that

$$\frac{1}{\mathcal{M}(\kappa)}\|(i\kappa I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall \kappa \in \mathbb{R}bmcfm$$

for a positive constant C. By applying Theorems 5 and 6, following the form of $\mathcal{M}$, we find the main result. □

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