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ON BOUNDARY VALUE PROBLEMS WITH IMPLICIT RANDOM NON-CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we present some results on the existence, uniqueness and Ulam stability for a class of problems for nonlinear implicit random fractional differential equations with non-conformable derivatives. For our proofs, we employ some suitable fixed point theorems. Finally, we provide some illustrative examples.

1. INTRODUCTION

Fractional calculus has emerged as a highly valuable tool in addressing the intricate structures encountered across numerous research disciplines. Its focus lies in extending the concepts of differentiation and integration, which traditionally apply to whole numbers, to non-integer orders. The theory and application of fractional calculus are substantial and have been extensively explored. For detailed information, we recommend referring to the following monographs [1–3, 23], and the following papers [7–9, 14, 19, 21]. Recently, numerous papers and monographs have been published, investigating a wide range of results pertaining to different forms of differential equations and inclusions with various types of conditions. Relevant works include [1, 17, 20, 22, 24, 25], along with the references cited within them.

In [11,18], the authors introduced a new conformable fractional derivative which obeys all the above-mentioned classical properties. It can be considered as a generalization of the conformable derivatives introduced by Khalil *et al.* [15].

In [16], the authors discussed the existence of solutions for the following initial value problem of conformable implicit impulsive fractional differential equations with infinite delay:

$$\begin{cases} \mathcal{T}^r_{\theta_k}\chi(\theta) = \Psi\left(\theta, \chi_\theta, \mathcal{T}^r_k\chi(\theta)\right), \quad \theta \in \Omega_k; k = 0, 1, \ldots, m, \\ \Delta\chi|_{\theta=\theta_k} = \hbar_k(\chi_{\theta_k^-}), \quad k = 1, 2, \ldots, m, \\ \chi(\theta) = \zeta(\theta), \quad \theta \in (-\infty, a], \end{cases}$$

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where $0 \leq a = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = \beta < \infty$, $\mathcal{T}^r_{\theta_k}\chi(\theta)$ is the conformable fractional derivative of order $0 < r < 1$, $\Psi : \Omega \times G \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [a, \beta]$, $\Omega_0 := [a, \theta_1]$, $\Omega_k := (\theta_k, \theta_{k+1}]; k = 1, 2, \ldots, m$, $\zeta : (-\infty, a] \to \mathbb{R}$ and $\hbar_k : G \to \mathbb{R}$ are given continuous functions. The results are based on the $\omega-\psi$-Geraghty type contraction and the fixed point theory.

In this paper, we study the existence and uniqueness of solutions for the implicit boundary value problem with nonlinear random fractional differential equation involving the non-conformable fractional derivative:

$$\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma, \wp) = \Im\left(\sigma, \xi(\sigma, \wp), \left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma, \wp), \wp\right);\ \sigma \in \Omega := [\kappa_1, \kappa_2], \wp \in \nabla, \tag{1.1}$$

$$a\xi(\kappa_1, \wp) + b\xi(\kappa_2, \wp) = c(\wp); \wp \in \nabla, \tag{1.2}$$

where $0 < \zeta < 1$, ${}^{N}_{\kappa_1}\mathcal{D}^{\zeta}$ is the non-conformable fractional derivative defined in [11, 18], $\Im : \Omega \times \mathbb{R}^2 \times \nabla \to \mathbb{R}$ is a given function to be specified later, $a, b \in \mathbb{R}$ are such that $a + b \neq 0$, $c : \nabla \to \mathbb{R}$ and ∇ is the sample space in a probability space.

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the non-conformable fractional derivatives used throughout this manuscript. In Section 3, we present two existence and uniqueness results for the problem (1.1)-(1.2) that are based on the Banach contraction principle and Itoh's fixed point theorem. Section 4 deals with the Ulam stability of our problem. In the last section, an illustrative example is provided in support of the obtained results.

2. Preliminaries

We denote by $\digamma := C(\Omega, \mathbb{R})$ the Banach space of all continuous functions from Ω into $\mathbb{R}$ with

$$\|\xi\|_\infty = \sup_{\sigma \in \Omega}\{|\xi(\sigma)|\}.$$

Consider the space $X^p_b(\kappa_1, \kappa_2)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions $\Im$ on $[\kappa_1, \kappa_2]$ for which $\|\Im\|_{X^p_b} < \infty$, where the norm is defined by:

$$\|\Im\|_{X^p_b} = \left(\int_{\kappa_1}^{\kappa_2} |\sigma^b \Im(\sigma)|^p \frac{d\sigma}{\sigma}\right)^{\frac{1}{p}},\ \ (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1 ([11, 18]). *Let $\Im : [a, +\infty) \longrightarrow \mathbb{R}$ be a given function, then the non-conformable fractional derivative of $\Im$ of order ζ is defined by*

$$\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\Im\right)(\sigma) = \lim_{\varepsilon \to 0} \frac{\Im\left(\sigma + \varepsilon e^{\sigma^{-\zeta}}\right) - \Im(\sigma)}{\varepsilon},$$

for $\sigma > 0$ and $\zeta \in (0, 1]$.

If $\Im$ is ζ-differentiable in some $(0,a)$, $a>0$, and $\lim_{\sigma\to 0+}\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\Im\right)(\sigma)$ *exists, then define* $\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\Im\right)(0)=\lim_{\sigma\to 0+}\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\Im\right)(\sigma)$. *If the non-conformable fractional derivative of $\Im$ of order ζ exists, then we simply say that $\Im$ is N-differentiable.*

Lemma 2.1 ([11, 18]). *If $\zeta\in[0,1]$, $\Im$ and g are two ζ-differentiable functions at a point σ and m,n are two given numbers, the non-conformable fractional derivative exhibits the following properties:*

- ${}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(j)=0$, *for any constant j;*
- ${}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(m\Im+ng)=m{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(\Im)+n{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(g)$;
- ${}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(\Im g)=g{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(\Im)+\Im{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(g)$;
- ${}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\left(\dfrac{\Im}{g}\right)=\dfrac{g{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(\Im)-\Im{}^{N}_{\kappa_1}\mathcal{D}^{\zeta}(g)}{g^2}$.

Definition 2.2 (The N-fractional integral [11, 18]). *For $\zeta\in(0,1]$ and a continuous function $\Im$, let*

$$\left({}^{N}I^{\zeta}_{a^+}\Im\right)(\sigma)=\int_a^{\sigma}\frac{\Im(s)}{e^{s^{-\zeta}}}ds.$$

Lemma 2.2 ([11, 18]). *If $\zeta\in(0,1]$, $\Im$ is N-differentiable function at a point σ, then we have:*

- $\left({}^{N}I^{\zeta}_{a^+}\,{}^{N}_{a}\mathcal{D}^{\zeta}(\Im)\right)(\sigma)=\Im(\sigma)-\Im(a)$;
- ${}^{N}_{a}\mathcal{D}^{\zeta}\left({}^{N}I^{\zeta}_{a^+}\Im\right)(\sigma)=\Im(\sigma)$.

Let $\beta_{\mathbb{R}}$ be the $\sigma-$algebra of Borel subsets of $\mathbb{R}$. A mapping $\gamma:\nabla\to\mathbb{R}$ is said to be measurable if for any $\Bbbk\in\beta_{\mathbb{R}}$, one has

$$\gamma^{-1}(\Bbbk)=\{\wp\subset\nabla:\gamma(\wp)\subset\Bbbk\}\subset A.$$

The following details and the properties of completely continuous random operators in Banach spaces are available in Itoh [13] and Engl [10].

Definition 2.3. *A mapping $\aleph:\nabla\times\mathbb{R}\to\mathbb{R}$ is called jointly measurable if for any $\Bbbk\subset\beta_{\mathbb{R}}$, one has*

$$\aleph^{-1}(\Bbbk)=\{(\wp,\gamma)\subset\nabla\times\mathbb{R}:\aleph(\wp,\gamma)\subset\Bbbk\}\subset A\times\beta_{\mathbb{R}},$$

where $A\times\beta_{\mathbb{R}}$ is the direct product of the $\sigma-$algebras A and $\beta_{\mathbb{R}}$ those defined in ∇ and $\mathbb{R}$ respectively.

Lemma 2.3. *Let $\aleph:\nabla\times\mathbb{R}\to\mathbb{R}$ be a mapping such that $\aleph(\cdot,\gamma)$ is measurable for all $\gamma\subset\mathbb{R}$, and $\aleph(\wp,\cdot)$ is continuous for all $\wp\subset\nabla$. Then the map $(\wp,\gamma)\to\aleph(\wp,\gamma)$ is jointly measurable.*

Definition 2.4. *A function $\Im:\Omega\times\mathbb{R}\times\nabla\to\mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:*

(i) *The map* $(\sigma, \wp) \to \Im(\sigma, \eta, \bar{\eta}, \wp)$ *is jointly measurable for all* $\eta \subset \mathbb{R}$, *and*
(ii) *The map* $\eta \to \Im(\sigma, \eta, \bar{\eta}, \wp)$ *is continuous for almost all* $\sigma \in \Omega$ *and* $\wp \subset \nabla$.

Definition 2.5. *Let* $\aleph : \nabla \times \mathbb{R} \to \mathbb{R}$ *be a mapping. Then* $\aleph$ *is called a random operator if* $\aleph(\wp, \eta)$ *is measurable in* $\wp$ *for all* $\eta \subset \mathbb{R}$ *and it is expressed as* $\aleph(\wp)\eta = \aleph(\wp, \eta)$. *In this case we also say that* $\aleph(\wp)$ *is a random operator on* $\mathbb{R}$. *A random operator* $\aleph(\wp)$ *on* $\mathbb{R}$ *is called continuous (resp. compact, totally bounded and completely continuous) if* $\aleph(\wp, \eta)$ *is continuous (resp. compact, totally bounded and completely continuous)in* η *for all* $\wp \subset \nabla$.

Definition 2.6. *Let* $\Lambda(\Theta)$ *be the family of all nonempty subsets of* Θ *and* $\mathfrak{Q}$ *be a mapping from* ∇ *into* $\Lambda(\Theta)$. *A mapping* $\aleph : \{(\wp, \eta) : \wp \subset \nabla, \xi \subset \mathfrak{Q}(\wp)\} \to \Theta$ *is called a random operator with stochastic domain* $\mathfrak{Q}$ *if* $\mathfrak{Q}$ *is measurable (i.e for all closed* $A \subset \Theta$, $\{\wp \subset \nabla, \mathfrak{Q}(\wp) \cap A \neq \emptyset\}$ *is measurable) and for all open* $D \subset \Theta$ *and all* $\eta \subset \Theta, \{\wp \subset \nabla : \eta \subset \mathfrak{Q}(\wp), \aleph(\wp, \eta) \subset D\}$ *is measurable.* $\aleph$ *will be called continuous if every* $\aleph(\wp)$ *is continuous. For a random operator* $\aleph$, *a mapping* $\eta : \nabla \to \Theta$ *is called random (stochastic) fixed point of* $\aleph$ *if for P-almost all* $\wp \subset \nabla, \eta(\wp) \subset \mathfrak{Q}(\wp)$ *and* $\aleph(\wp)\eta(\wp) = \eta(\wp)$ *and for all open* $D \subset \Theta, \{\wp \subset \nabla : \eta(\wp) \subset D\}$ *is measurable.*

Lemma 2.4. *Let* $h : \Omega \to \mathbb{R}$ *be a continuous function. A function* $\xi \in \digamma$ *is a solution of the problem*

$$\begin{cases} \left({}^N_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma) = h(\sigma);\ \sigma \in \Omega := [\kappa_1, \kappa_2], \\ a\xi(\kappa_1) + b\xi(\kappa_2) = c, \end{cases} \tag{2.1}$$

where $a, b \in \mathbb{R},\ a + b \neq 0$, *with* $c \in \mathbb{R}$ *if and only if* ξ *satisfies the following integral equation*

$$\xi(\sigma) = C_0 + \int_{\kappa_1}^{\sigma} \frac{h(s)}{e^{s-\zeta}} ds, \tag{2.2}$$

where

$$C_0 = \frac{1}{a+b}\left[c - a\left({}^N I^{\zeta}_{a^+} h\right)(\kappa_1) - b\left({}^N I^{\zeta}_{a^+} h\right)(\kappa_2)\right].$$

Proof. From Lemma 2.2 and by applying the operator $I^{\zeta}_{a^+}(\cdot)$ on equation (2.1), we have

$$\xi(\sigma) - \xi(\kappa_1) = \left({}^N I^{\zeta}_{a^+} h\right)(\sigma).$$

From the condition (1.2), we get

$$a\xi(\kappa_1) + b\xi(\kappa_2) = a\left({}^N I^{\zeta} h\right)(\kappa_1) + a\xi(\kappa_1) + \left({}^N I^{\zeta} h\right)(\kappa_2) + b\xi(\kappa_2) = c,$$

thus we can obtain (2.2). The converse is easily demonstrated by employing Lemma 2.2. $\square$

3. Existence and Uniqueness Results

Lemma 3.1. *By a random solution of problem* (1.1)-(1.2)*, we mean a function* $\xi \in \digamma$ *that satisfies the equation*

$$\xi(\sigma,\wp) = C_0(\wp) + \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma,\xi(\sigma,\wp),\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp),\wp\right)}{e^{s-\zeta}}ds,$$

where

$$C_0(\wp) = \frac{1}{a+b}\left[c(\wp) - a\left({}^{N}I^{\zeta}_{a^+}\Im\right)(\kappa_1,\wp) - b\left({}^{N}I^{\zeta}_{a^+}\Im\right)(\kappa_2,\wp)\right].$$

In the sequel, the following hypotheses are used:

(H_1)**:** The function $\Im : \Omega\times\mathbb{R}^2\times\nabla \to \mathbb{R}$ is random Carathéodory.

(H_2)**:** There exist measurable functions $p_1, p_2 : \Omega \to L^{\infty}(\nabla,\mathbb{R}_+)$,

$$|\Im(\sigma,\eta_1,\bar{\eta_1},\wp) - \Im(\sigma,\eta_2,\bar{\eta_2},\wp)| \leq p_1(\sigma,\wp)|\eta_1-\eta_2| + p_2(\sigma,\wp)|\bar{\eta_1}-\bar{\eta_2}|,$$

for $\sigma\in\Omega$ and $\eta_1,\eta_2\in\mathbb{R}$, and $\bar{\beta_1},\bar{\beta_2}\in\mathbb{R}$, with

$$p_1^*(\wp) = \sup_{\sigma\in\Omega} p(\sigma,\wp) \ \text{ and } \ p_2^*(\wp) = \sup_{\sigma\in\Omega} p_2(\sigma,\wp) < 1.$$

(H_3)**:** There exist measurable functions $l_i : \Omega \longrightarrow L^{\infty}(\nabla,\mathbb{R}_+), i\in[1,3]$ such that

$$|\Im(\sigma,\eta,\bar{\eta},\wp)| \leq l_1(\sigma,\wp) + l_2(\sigma,\wp)\frac{|\eta|}{1+|\eta|} + l_3(\sigma,\wp)|\bar{\eta}|,$$

for $\sigma\in\theta$, $\eta\in\mathbb{R}$, $\bar{\eta}\in\mathbb{R}$ and $\wp\in\nabla$.
Set

$$l_i^*(\wp) = \sup_{\sigma\in\Omega} l_i(\sigma,\wp), \quad \text{and } \ l_3^*(\wp) = \sup_{\sigma\in\Omega} l_3(\sigma,\wp) < 1.$$

Now we declare and demonstrate our first existence result for problem (1.1)-(1.2) which is based on the Banach contraction principle [12].

Theorem 3.1. *Assume that* (H_1)-(H_2) *hold. If*

$$\ell := \frac{p_1^*(\wp)(\kappa_2-\kappa_1)}{1-p_2^*(\wp)} < 1, \tag{3.1}$$

then the problem (1.1)-(1.2) *has a unique solution.*

Proof. Let $\aleph : \digamma\times\nabla \to \digamma$ be the operator defined by

$$(\aleph\xi)(\sigma,\wp) = C_0(\wp) + \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma,\xi(\sigma,\wp),\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp),\wp\right)}{e^{s-\zeta}}ds, \tag{3.2}$$

where G is a function satisfying the following functional equation

$$G(\sigma,\wp) = \Im\left(\sigma,\xi(\cdot,\wp),G(\sigma,\wp),\wp\right).$$

Let $\xi_1, \xi_2 \in \digamma$, then we have

$$|(\aleph\xi_1)(\sigma,\wp)-(\aleph\xi_2)(\sigma,\wp)| \leq \int_{\kappa_1}^{\sigma} \frac{|G_1(s,\wp)-G_2(s,\wp)|}{e^{s^{-\zeta}}}ds, \tag{3.3}$$

where G_1, G_2 are the functions satisfying the following functional equations

$$\begin{aligned} G_1(\sigma,\wp) &= \Im\left(\sigma,\xi_1(\sigma,\wp),G_1(\sigma,\wp),\wp\right),\\ G_2(\sigma,\wp) &= \Im\left(\sigma,\xi_2(\sigma,\wp),G_2(\sigma,\wp),\wp\right). \end{aligned}$$

By (H_2), we have

$$\begin{aligned} &|G_1(\sigma,\wp)-G_2(\sigma,\wp)|\\ &\quad = |\Im\left(\sigma,\xi_1(\sigma,\wp),G_1(\sigma,\wp),\wp\right)-\Im\left(\sigma,\xi_2(\sigma,\wp),G_2(\sigma,\wp),\wp\right)|\\ &\quad \leq p_1(\sigma,\wp)|\xi_1(\sigma,\wp)-\xi_2(\sigma,\wp)|+p_2(\sigma,\wp)|G_1(\sigma,\wp)-G_2(\sigma,\wp)|\\ &\quad \leq p_1^*(\wp)|\xi_1(\sigma,\wp)-\xi_2(\sigma,\wp)|+p_2^*(\wp)|G_1(\sigma,\wp)-G_2(\sigma,\wp)|, \end{aligned}$$

thus

$$|G_1(\sigma,\wp)-G_2(\sigma,\wp)| \leq \frac{p_1^*(\wp)}{1-p_2^*(\wp)}|\xi_1(\sigma,\wp)-\xi_2(\sigma,\wp)|.$$

We may obtain now that

$$\begin{aligned} |(\aleph\xi_1)(\sigma,\wp)-(\aleph\xi_2)(\sigma,\wp)| &\leq \int_{\kappa_1}^{\sigma} \frac{\frac{p_1^*(\wp)}{1-p_2^*(\wp)}}{e^{s^{-\zeta}}}|\xi_1(s,\wp)-\xi_2(s,\wp)|ds\\ &\leq \frac{p_1^*(\wp)(\kappa_2-\kappa_1)}{1-p_2^*(\wp)}\|\xi_1-\xi_2\|_\infty\\ &\leq \ell\|\xi_1-\xi_2\|_\infty. \end{aligned}$$

Thus

$$\|\aleph\xi_1-\aleph\xi_2\|_\infty \leq \ell\|\xi_1-\xi_2\|_\infty.$$

Hence, by the Banach contraction principle, $\aleph$ has a unique fixed point which is a unique random solution of the problem (1.1)-(1.2). $\square$

Now, we prove an existence result for the problem (1.1)-(1.2) based on Itoh's fixed point theorem [13].

Theorem 3.2. *Assume that* (H_1)-(H_3) *hold. Then, the problem* (1.1)-(1.2) *has at least one random solution.*

Proof. The function $\Im$ is absolutely continuous for all $\wp \in \nabla$ and $\sigma \in \Omega$, so then ξ is a random solution for (1.1)-(1.2) if and only if $\xi = (\aleph\xi)(\sigma,\wp)$.

Set

$$\beth(\wp) > \frac{|C_0(\wp)| + l_1^*(\wp)\Psi}{1 - l_2^*(\wp)\Psi}, \quad \wp \in \nabla, \tag{3.4}$$

where

$$\Psi = \frac{(\kappa_2 - \kappa_1)}{1 - l_3^*(\wp)}.$$

Define the ball

$$\Bbbk_{\beth(\wp)} := \{\xi \in \digamma : \|\xi\| \leq \beth(\wp)\}.$$

For any $\wp \in \nabla$ and each $\sigma \in \Omega$, we have

$$|(\aleph\xi)(\sigma,\wp)| \leq |C_0(\wp)| + \int_{\kappa_1}^{\sigma} \frac{|G(\sigma,\wp)|}{e^{s^{-\zeta}}} ds. \tag{3.5}$$

By the hypothesis (H_3), for $\sigma \in \theta$, we have

$$\begin{aligned}|G(\sigma,\wp)| &= |\Im(\sigma, \xi(\sigma,\wp), G(\sigma,\wp), \wp)| \\ &\leq l_1(\sigma,\wp) + l_2(\sigma,\wp)|\xi(\sigma,\wp)| + l_3(\sigma,\wp)|G(\sigma,\wp)|,\end{aligned}$$

which implies that

$$|G(\sigma,\wp)| \leq l_1^*(\wp) + l_2^*(\wp)|\xi(\sigma,\wp)| + l_3^*(\wp)|G(\sigma,\wp)|,$$

and then

$$|G(\sigma,\wp)| \leq \frac{l_1^*(\wp) + l_2^*(\wp)\beth(\wp)}{1 - l_3^*(\wp)}.$$

Thus, for $\sigma \in \Omega$ and from (3.5), we obtain

$$\begin{aligned}|(Tx)(\sigma,\wp)| &\leq |C_0(\wp)| + \frac{(\kappa_2 - \kappa_1)}{1 - l_3^*(\wp)}(l_1^*(\wp) + l_2^*(\wp)\beth(\wp)) \\ &\leq \beth(\wp).\end{aligned}$$

This shows that $\aleph(\wp)$ transforms the ball $\Bbbk_{\beth(\wp)}$ into itself. We will demonstrate that $\aleph : \nabla \times \Bbbk_{\beth(\wp)} \to \Bbbk_{\beth(\wp)}$ verifies all the requirements of Itoh's random fixed point theorem [13].

Step 1. $\aleph(\wp)$ is a random operator.
Since $\Im(\sigma,\xi,\gamma,\wp)$ is random Carathéodory, the map $\wp \longrightarrow \Im(\sigma,\eta,\wp)$ is measurable in view of Definition 2.6 and further the integral is a limit of a finite sum of measurable functions so therefore the map

$$\wp \quad \mapsto \quad C_0(\wp) + \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds$$

is measurable. As a result, $\aleph(\wp)$ is a random operator.

Step 2. The operator $\aleph$ is continuous and bounded.
Let $\{\xi_n\}$ be a sequence such that $\xi_n \longrightarrow x$ in $\digamma$. We have

$$|(\aleph\xi_n)(\sigma,\wp)-(\aleph\xi)(\sigma,\wp)| \leq \int_{\kappa_1}^{\sigma} \frac{|G_n(s,\wp)-G(s,\wp)|ds}{e^{s-\zeta}}ds,$$

where

$$G_n(\sigma,\wp)=\Im\left(\sigma,\xi_n(\sigma,\wp),G_n(\sigma,\wp),\wp\right),$$

and

$$G(\sigma,\wp)=\Im\left(\sigma,\xi(\sigma,\wp),G(\sigma,\wp),\wp\right).$$

Since $\xi_n \longrightarrow \xi$, and by (H_1), we get $G_n(\sigma,\wp) \longrightarrow G(\sigma,\wp)$ as $n \longrightarrow \infty$ for each $\sigma \in I$. Then, by the Lebesgue dominated convergence theorem and (H_1), we get

$$\|\aleph(\xi_n)-\aleph(\xi)\|_\infty \longrightarrow 0 \; as \; n \longrightarrow \infty.$$

Consequently, since $\aleph(\wp)$ is a continuous random operator with stochastic domain, we may conclude also that $\aleph(\Bbbk_{\daleth(\wp)}) \subset \Bbbk_{\daleth(\wp)}$.

Step 3. $\aleph(\Bbbk_{\daleth(\wp)})$ is equicontinuous.
For $\kappa_1 \leq \sigma_1 \leq \sigma_2 \leq \kappa_2$, and $\xi \in \Bbbk_{\daleth(\wp)}$, we have

$$\begin{aligned}
&|(\aleph\xi)(\sigma_2,\wp)-(\aleph\xi)(\sigma_1,\wp)| \\
&\quad\leq \left|\int_{\kappa_1}^{\sigma_2} \frac{G(\sigma_2,\wp)}{e^{s-\zeta}}ds - \int_{\kappa_1}^{\sigma_1} \frac{G(\sigma_1,\wp)}{e^{s-\zeta}}ds\right| \\
&\quad\leq \left|\int_{\sigma_1}^{\sigma_2} \frac{G(\sigma_2,\wp)}{e^{s-\zeta}}ds - \int_{\kappa_1}^{\sigma_1} \frac{G(\sigma_1,\wp)}{e^{s-\zeta}}ds + \int_{\kappa_1}^{\sigma_1} \frac{G(\sigma_2,\wp)}{e^{s-\zeta}}ds\right| \\
&\quad\leq \frac{(\sigma_2-\sigma_1)}{1-l_3^*(\wp)}(l_1^*(\wp)+l_2^*(\wp)\daleth(\wp)).
\end{aligned}$$

As $\sigma_1 \longrightarrow \sigma_2$ the right hand side of the above inequality tends to zero. As a consequence of Step 1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that $\aleph$ is continuous and completely continuous. From an application of Itoh's random fixed point theorem [13], we conclude that $\aleph$ has a fixed point which is a random solution of the problem (1.1)-(1.2). □

4. ULAM STABILITY RESULTS

Now, we consider the Ulam stability for the problem (1.1)-(1.2). Let $\varepsilon > 0$ and $\Xi : \Omega \times \nabla \to \mathbb{R}_+$ be a measurable function. We consider the following inequalities

$$\left|\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp)-\Im\left(\sigma,\xi(\sigma,\wp),\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp),\wp\right)\right| \leq \varepsilon;\ \sigma\in\Omega,\ \wp\in\nabla. \quad (4.1)$$

$$\left|\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp)-\Im\left(\sigma,\xi(\sigma,\wp),\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp),\wp\right)\right| \leq \Xi(\sigma,\wp);\ \sigma\in\Omega,\ \wp\in\nabla. \quad (4.2)$$

$$\left|\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp)-\Im\left(\sigma,\xi(\sigma,\wp),\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp),\wp\right)\right| \leq \varepsilon\Xi(\sigma,\wp);\ \sigma\in\Omega,\ \wp\in\nabla. \quad (4.3)$$

Definition 4.1 ([3]). *The problem* (1.1)-(1.2) *is Ulam-Hyers stable if there exists a real number* $\delta_{\Im} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\xi(\cdot, \wp) \in \mathcal{F}$ *of the inequality (4.1), there exists a solution* $\gamma \in \mathcal{F}$ *of* (1.1)-(1.2) *with*

$$|\xi(\sigma, \wp) - \gamma(\sigma, \wp)| \leq \varepsilon \delta_{\Im};\ \sigma \in \Omega.$$

Definition 4.2 ([3]). *The problem* (1.1)-(1.2) *is generalized Ulam-Hyers stable if there exists* $\delta_{\Im} \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\delta_{\Im}(0) = 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\xi(\wp) \in \mathcal{F}$ *of the inequality (4.1), there exists a solution* $\gamma \in \mathcal{F}$ *of* (1.1)-(1.2) *with*

$$|\xi(\sigma, \wp) - \gamma(\sigma, \wp)| \leq \delta_{\Im}(\varepsilon);\ \sigma \in \Omega.$$

Definition 4.3 ([3]). *The problem* (1.1)-(1.2) *is Ulam-Hyers-Rassias stable with respect to* Ξ *if there exists a real number* $\delta_{\Im,\Xi} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\xi(\wp) \in \mathcal{F}$ *of the inequality (4.3), there exists a solution* $\gamma \in \mathcal{F}$ *of* (1.1)-(1.2) *with*

$$|\xi(\sigma, \wp) - \gamma(\sigma, \wp)| \leq \varepsilon \delta_{\Im,\Xi} \Xi(\sigma, \wp);\ \sigma \in \Omega.$$

Definition 4.4 ([3]). *The problem* (1.1)-(1.2) *is generalized Ulam-Hyers-Rassias stable with respect to* Ξ *if there exists a real number* $\delta_{\Im,\Xi} > 0$ *such that for each solution* $\xi \in \mathcal{F}$ *of the inequality (4.2), there exists a solution* $\gamma \in \mathcal{F}$ *of* (1.1)-(1.2) *with*

$$|\xi(\sigma, \wp) - \gamma(\sigma, \wp)| \leq \delta_{\Im,\Xi} \Xi(\sigma, \wp);\ \sigma \in \Omega.$$

Remark 4.1. A function $\xi(\cdot, \wp) \in \mathcal{F}$ is a solution of the inequality (4.2) if and only if there exist a function $g(\cdot, \wp) \in \mathcal{F}$ (which depends on ξ) such that

$$|g(\sigma, \wp)| \leq \Xi(\sigma, \wp),$$

and

$$\begin{aligned}\left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma, \wp) = \Im\left(\sigma, \xi(\sigma, \wp), \left({}^{N}_{\kappa_1}\mathcal{D}^{\zeta}\xi\right)(\sigma, \wp), \wp\right) \\ + g(\sigma, \wp);\ \text{for } \sigma \in \theta,\ \text{and } \wp \in \nabla.\end{aligned}$$

The following hypotheses will be used in the sequel.

(H_4) Let $\Xi(\cdot, \wp) \in L^1(\mathbb{R}_+)$, and there exists a constant $j_{\Xi} > 0$, such that for any $\wp \in \nabla$, and each $\sigma \in \Omega$ we have

$$\left({}^{N}I^{\zeta}_{a^+}\Xi\right)(\sigma, \wp)ds \leq j_{\Xi}\Xi(\sigma, \wp).$$

(H_5) There exist measurable functions $l_i : \Omega \longrightarrow L^{\infty}(\nabla, \mathbb{R}_+); i = 1, 2, 3$ such that

$$|\Im(\sigma, \eta, \bar{\eta}, \wp)| \leq l_1(\sigma, \wp)\Xi(\sigma, \wp) + l_2(\sigma, \wp)\Xi(\sigma, \wp)\frac{|\eta|}{1 + |\eta|} + l_3(\sigma, \wp)|\bar{\eta}|,$$

for $\sigma \in \theta$, $\eta \in \mathbb{R}$, $\bar{\eta} \in \mathbb{R}$ and $\wp \in \nabla$.
Set

$$l_i^*(\wp) = \sup_{\sigma \in \theta} l_i(\sigma, \wp), \quad \text{and} \quad l_3^*(\wp) = \sup_{\sigma \in \Omega} l_3(\sigma, \wp) < 1.$$

Theorem 4.1. *Assume that the hypotheses* (H_1)-(H_5) *hold. Then the problem* (1.1)-(1.2) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $\xi \in \digamma$ be a solution of the inequality (4.2). By Remark 4.1, for any $\wp \in \nabla$ and each $\sigma \in \Omega$, we have

$$\xi(\sigma,\wp) = C_0(\wp) + \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right) + g(s,\wp)}{e^{s^{-\zeta}}} ds.$$

Then ξ is a solution of the following integral inequality

$$|\xi(\sigma,\wp) - C_0(\wp) - \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds| \leq \left({}^{N}I_{a^+}^{\zeta}\Xi\right)(\sigma,\wp)ds. \tag{4.4}$$

Thus by (H_3), we obtain

$$|\xi(\sigma,\wp) - C_0(\wp) - \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds| \leq j_{\Xi}\Xi(\sigma,\wp).$$

There exists a random solution γ of the random problem (1.1)-(1.2). That is

$$\gamma(\sigma,\wp) = C_0(\wp) + \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \gamma(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\gamma\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds,$$

where

$$H(\sigma,\wp) = \Im\left(\sigma, \gamma(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\gamma\right)(\sigma,\wp), \wp\right).$$

Let ξ be a solution of the inequality (4.2), for any $\wp \in \nabla$ and each $\sigma \in \Omega$, we have

$$\begin{aligned}
|\xi(\sigma,\wp) - \gamma(\sigma,\wp)| \leq & \Bigg| \xi(\sigma,\wp) - C_0(\wp) - \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds \\
& + {\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \xi(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\xi\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds \\
& - \int_{\kappa_1}^{\sigma} \frac{\Im\left(\sigma, \gamma(\sigma,\wp), \left({}_{\kappa_1}^{N}\mathcal{D}^{\zeta}\gamma\right)(\sigma,\wp), \wp\right)}{e^{s^{-\zeta}}} ds \Bigg| \\
\leq & \; j_{\Xi}\Xi(\sigma,\wp) + \int_{\kappa_1}^{\sigma} \frac{|G(\sigma,\wp)|}{e^{s^{-\zeta}}} ds + \int_{\kappa_1}^{\sigma} \frac{|H(\sigma,\wp)|}{e^{s^{-\zeta}}} ds.
\end{aligned}$$

By the hypotheses (H_4) and (H_5), for $\sigma \in \Omega$ we have

$$|\xi(\sigma,\wp) - \gamma(\sigma,\wp)| \leq j_{\Xi}\Xi(\sigma,\wp) + 2\frac{l_1^*(\wp) + l_2^*(\wp)}{1 - l_3^*(\wp)} \left({}^{N}I_{a^+}^{\zeta}\Xi\right)(s,\wp)ds$$

$$
\begin{aligned}
&\leq j_{\Xi}\Xi(\sigma,\wp)\left[1+2\frac{l_1^*(\wp)+l_2^*(\wp)}{1-l_3^*(\wp)}\right]\\
&\leq \delta_{\mathfrak{I},\Xi}\Xi(\sigma,\wp).
\end{aligned}
$$

Hence, the problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable. □

5. EXAMPLES

Example 5.1. *We equip the space* $\mathbb{R}_-^* := (-\infty, 0)$ *with the standard* σ*-algebra, which consists of Lebesgue measurable subsets of* $\mathbb{R}_-^*$. *Now, we consider the following example of our problem* (1.1)-(1.2)*:*

$$
\begin{cases}
\left({}_{0}^{N}\mathcal{D}^{\frac{1}{2}}\xi\right)(\sigma,\wp)=\mathfrak{I}\left(\sigma,\xi(\sigma,\wp),\left({}_{\kappa_1}^{N}\mathcal{D}^{\frac{1}{2}}\xi\right)(\sigma,\wp),\wp\right);\ \sigma\in\Omega:=[0,1],\wp\in\nabla,\\
\xi(\kappa_1,\wp)+\xi(\kappa_2,\wp)=c(\wp);\wp\in\nabla.
\end{cases}
\tag{5.1}
$$

Set

$$
\mathfrak{I}(\sigma,\beta_1,\beta_2,\wp)=\frac{\cos(\sigma)(|\beta_1|+|\beta_2|)}{164e^{\sigma+5}(|\wp|+1)(1+|\beta_1|+|\beta_2|)},
$$

where $\zeta=\frac{1}{2}$, $\kappa_1=0$, $\kappa_2=1$, $a=b=1$, $\beta_1,\beta_2\in\mathbb{R}$.
It is clear that the function $\mathfrak{I}$ *verifies the hypothesis* (H_1), *and for each* $\beta_1,\bar{\beta_1},\beta_2,\bar{\beta_2}\in\mathbb{R}$ *and* $\sigma\in[0,1]$, *we have*

$$
|\mathfrak{I}(\sigma,\beta_1,\beta_2,\wp)-\mathfrak{I}(\sigma,\bar{\beta_1},\bar{\beta_2},\wp)|\leq\frac{\cos(\sigma)}{164e^{\sigma+5}(|\wp|+1)}\left[|\beta_1-\bar{\beta_1}|+|\beta_2-\bar{\beta_2}|\right].
$$

Therefore, (H_2) *is verified with*

$$
p_1(\sigma,\wp)=p_2(\sigma,\wp)=\frac{\cos(\sigma)}{164e^{\sigma+5}(|\wp|+1)},
$$

and

$$
p_1^*(\wp)=p_2^*(\wp)=\frac{1}{164e^{5}(|\wp|+1)}.
$$

Also, for $\sigma\in[0,1]$ *we have*

$$
\begin{aligned}
\ell &:= \frac{p_1^*(\wp)(\kappa_2-\kappa_1)}{1-p_2^*(\wp)}\\
&= \frac{1}{164e^{5}(|\wp|+1)-1}\\
&< 1.
\end{aligned}
$$

Then, the condition (3.1) *is satisfied. Hence, as all conditions of Theorem 3.1 are met, the problem* (5.1) *admits a unique random solution. Also, the hypotheses* (H_4) *and* (H_5) *are satisfied with*

$$\Xi(\sigma,\wp)=4\sqrt{\pi}\ \ and\ \ l_i(\sigma,\wp)=\frac{\cos(\sigma)}{164e^{\sigma+5}(|\wp|+1)};\ i=1,2,3.$$

Hence, Theorem 4.1 implies that the problem (5.1) is generalized Ulam-Hyers-Rassias stable.

REFERENCES

[1] S. Abbas, M. Benchohra, J. R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations*: Existence and Stability, de Gruyter, Berlin, 2018.

[2] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in Fractional Differential Equations*. Springer, New York, 2012.

[3] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] T. Abdeljawad, On conformable fractional calculus, *Journal of Computational and Applied Mathematics.* **279** (2015), 57-66.

[5] F. Bekada, S. Abbas, and M. Benchohra, *Boundary value problem for Caputo-Fabrizio random fractional differential equations*, Moroccan J. Pure Appl. Anal. (MJPAA) 6 (2) (2020), 218-230.

[6] F. Bekada, S. Abbas, M. Benchohra, and J. J. *Nieto, Dynamics and stability for Katugampola random fractional differential equations*, AIMS Mathematics. (2021), 8654-8666.

[7] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra and C. Tunc, *Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses*, Appl. Anal. Optim. **6** (2022), 79-94.

[8] M. Benchohra, F. Bouazzaoui, E. Karapinar and A. Salim, *Controllability of second order functional random differential equations with delay*, Math. **10** (2022), 16pp. https://doi.org/10.3390/math10071120

[9] C. Derbazi, H. Hammouche, A. Salim and M. Benchohra, *Measure of noncompactness and fractional Hybrid differential equations with Hybrid conditions*, Differ. Equ. Appl. **14** (2022), 145-161. http://dx.doi.org/10.7153/dea-2022-14-09

[10] H. W. Engl, *A general stochastic fixed-point theorem for continuous random operators on stochastic domains*, Anal. Appl. **66** (1978), 220-231.

[11] P. M. Guzman, G. Langton, L. L. Motta, J. Medina and J. E. Napoles V., *A New definition of a fractional derivative of local type*, *J. Math. Anal.* **9** (2018), 88-98.

[12] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[13] S. Itoh, R*andom fixed point theorems with applications to random differential equations in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 261-273.

[14] A. Heris, A. Salim, M. Benchohra and E. Karapinar, *Fractional partial random differential equations with infinite delay*, Results in Physics. (2022). https://doi.org/10.1016/j.rinp.2022.105557

[15] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, **264** (2014), 65-70.

[16] S. Krim, A. Salim, S. Abbas and M. Benchohra, *On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces*, Rend. Circ. Mat. Palermo (2). (2022), 1-14. https://doi.org/10.1007/s12215-022-00818-8

[17] N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, *On implicit fractional q-difference equations: Analysis and stability*, Math. Methods Appl. Sci. **45** (17) (2022), 10775-10797. https://doi.org/10.1002/mma.8417

[18] J. E. Napoles V., P. M. Guzman and L. M. Lugo, *Some new results on nonconformable fractional calculus*, Advances in Dynamical Systems and Applications, **13** (2018), 167-175.

[19] A. Salim, S. Abbas, M. Benchohra and E. Karapinar, *Global stability results for Volterra-Hadamard random partial fractional integral equations*, Rend. Circ. Mat. Palermo (2), (2022), 1-13. https://doi.org/10.1007/s12215-022-00770-7

[20] A. Salim, M. Benchohra, J. R. Graef and J. E. Lazreg, *Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations*, J. Fixed Point Theory Appl. **24** (2022), 14 pp. https://doi.org/10.1007/s11784-021-00920-x

[21] A. Salim, J. E. Lazreg, B. Ahmad, M. Benchohra and J. J. Nieto, *A study on k-generalized ψ-Hilfer derivative operator*, Vietnam J. Math. (2022).

[22] A. Salim, F. Mesri, M. Benchohra and C. Tunç, *Controllability of second order semilinear random differential equations in Fréchet spaces*, Mediterr. J. Math. **20** (84) (2023), 1-12. https://doi.org/10.1007/s00009-023-02299-0

[23] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[24] O. Tunç and C. Tunç, *Solution estimates to Caputo proportional fractional derivative delay integro-differential equations*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117** (2023), 12. https://doi.org/10.1007/s13398-022-01345-y

[25] O. Tunç and C. Tunç, *Ulam stabilities of nonlinear iterative integro-differential equations*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117** (2023), 118. https://doi.org/10.1007/s13398-023-01450-6

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